ADVANCES IN MATHEMATICS 14, 271–308 (1974)

On the Virtual Groups Defined by Ergodic Actions of $\mathbb{R}^n$ and $\mathbb{Z}^n$ *

PETER FORREST

The University of Western Australia, Nedlands, Western Australia 6009

INTRODUCTION

Warren Ambrose made a significant discovery in Ergodic Theory in his paper "Representations of Ergodic Flows" [6]. In this paper he shows that every nontrivial ergodic action of the real line on a standard measure space with an invariant measure can be represented as the "flow built under a function" from an ergodic action of the integers. If we replace the condition that the measure be invariant by the weaker condition that it be quasi-invariant, then we can formulate this result abstractly, using George W. Mackey's concept of the virtual group [7]. The theorem now states that proper virtual subgroups of the real line can be embedded in the integers.

R. M. Belinskaya has shown in her paper "Partition of Lebesque Space in Trajectories Defined by Ergodic Automorphism" [8] that the equivalence relations defined by proper ergodic actions of the integers (on standard measure spaces with invariant measures) are all isomorphic. This implies the weaker theorem that the corresponding virtual groups are isomorphic.

In this thesis, I free these results from their dependence on the ordering of $\mathbb{R}$ and $\mathbb{Z}$ and so am able to generalize them to $\mathbb{R}^m \times \mathbb{Z}^n$.

Consider a proper free ergodic action of $\mathbb{R}^m \times \mathbb{Z}^n$ on (S, C), a standard space with an invariant measure class. My first result (Theorem 1) is that the virtual subgroup of $\mathbb{R}^m \times \mathbb{Z}^n$ so defined can be embedded in Ω, a countable direct sum of groups of order two. Using this I can embed the virtual group in the integers (Theorem 2). This is the appropriate

* This paper was presented as a thesis to the Department of Mathematics, Harvard University, Cambridge, Massachusetts in 1972 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

abstract generalization of Warren Ambrose's theorem. Not much more can be said in this general case (see Proposition 6.3).

However, if C contains an invariant probability measure, this virtual group is always the same (Theorem 3), that is, it can be exhibited by some particular action of the integers chosen once and for all. If $m = 0$, we can generalize R. M. Belinskaya's theorem, namely, any two free, ergodic actions of $\mathbb{Z}^n$ and $\mathbb{Z}^k$ respectively define isomorphic equivalence relations (Theorem 4).

In Theorems 5 and 6, I interpret these results concretely. Theorem 5 generalizes Warren Ambrose's result. Theorem 6 states that every proper free ergodic action of $\mathbb{R}^m \times \mathbb{Z}^n$ is a projective limit of certain rather simple actions.

The possibility that these actions are not free presents no difficulty, as is seen in Proposition 7.1.

To some extent this work overlaps that of H. A. Dye on groups of measure-preserving transformations [11, 12]. Given two countable groups of measure-preserving transformations acting ergodically, his concept of equivalence implies the isomorphism of the corresponding ergodic equivalence relations. From Corollary 4.1 of [12, p. 561] and Theorem Three of [11, p. 151], all type II abelian groups of measure-preserving transformations are equivalent in H. A. Dye's sense.

Thus the generalization of R. M. Belinskaya's theorem to type II countable abelian groups follows from H. A. Dye's work.

I wish to acknowledge my debt to George W. Mackey, who not only introduced me to the work of Warren Ambrose, R. M. Belinskaya, and himself but also greatly encouraged me to perservere in this research. I also wish to thank Arlan Ramsay for reading this paper and making some helpful suggestions.

I. A Survey of the Theory of Virtual Groups

1.1. *Ergodicity*

Let Γ be a locally compact separable group. Let S be a Borel Γ-space (see [1, p. 328]). Let μ be a finite quasi-invariant measure on S. The action of Γ on (S, μ) is said to be ergodic if, given any Γ-invariant Borel set $T \subseteq S$, either $\mu(T) = 0$ or $\mu(S - T) = 0$.

If Λ is a closed subgroup of Γ and $S = \Gamma/\Lambda$, any invariant Borel set in S is $\varnothing$ or S. Thus, if μ is a quasi-invariant measure on S, (S, μ)

is an ergodic Γ-space. This is said to be the transitive case. An action of Γ on (S, μ) is said to be proper ergodic if there is no Γ-orbit T such that $\mu(S - T) = 0$.

An important property of proper ergodic actions of Γ is that there is no Borel cross-section for the orbits of Γ.

A more algebraic approach to ergodicity is to consider Boolean Γ-spaces, i.e., Borel representations of Γ, as automorphisms of the standard σ-Boolean algebra **B** (see [1, p. 328]). Borel Γ-spaces with quasi-invariant measures give rise to Boolean Γ-spaces (we obtain a Boolean Γ-space by considering Borel sets modulo null sets). In [1], George W. Mackey shows that every Boolean Γ-space arises in this way. Ergodic actions of Γ correspond to irreducible representations of Γ as automorphisms of **B**.

The automorphisms of **B** can be identified with the automorphisms of the commutative von Neumann algebra $L^\infty([0, 1]$, Lebesgue). Ergodic actions of Γ are thus just particular cases of irreducible representations of Γ as automorphisms of a von Neumann algebra.

Another related concept of ergodicity is the ergodic equivalence relation. If we ignore certain technicalities, we can define an ergodic equivalence relation on (S, μ), a standard space with a finite measure, to be an equivalence relation such that any Borel set composed of equivalence classes is either null or conull.

If Γ acts ergodically on (S, μ), the orbits of Γ define the equivalence classes of an ergodic equivalence relation.

1.2. *The Virtual Approach to Ergodicity*

Suppose we aim to classify all ergodic actions of Γ up to isomorphism. In the transitive case, we have first to find all subgroups Λ of Γ and then find all possible imbeddings of Λ in Γ. George W. Mackey has found an analog of the subgroup Λ, which he calls a virtual subgroup. Thus to classify ergodic actions, we have first to find all virtual groups which can be imbedded in Γ and then to find all such imbeddings. In the transitive case, the virtual group is just the isomorphism class of Λ.

Consider an ergodic equivalence relation. Any equivalence relation can be thought of as a groupoid, so we are lead to the concept of an ergodic groupoid (defined below).

We can think of a groupoid as a category and a homomorphism of groupoids as a functor. The concept of the natural equivalence of two functors corresponds to the concept of the similarity of two homomorphisms of ergodic groupoids. To obtain the category of Virtual Groups, we identify similar homomorphisms. This forces an identification of certain ergodic groupoids (cf. homotopy equivalence: by identifying homotopic maps we obtain the new category Hot from the category Top).

We will now define an ergodic groupoid.

DEFINITION 1.3. A groupoid G is said to be analytic if it is an analytic space and

(a) the set $G^{(2)}$ of composable elements is Borel; ($G^{(2)} = \{(x, y): xy \text{ is defined}\}$.)

(b) The maps $x \to x^{-1}$ and $(x, y) \to xy$ are Borel.

1.4 If G is an analytic groupoid, the units U form a Borel set. We identify units with the objects of the category. Also, we have Borel maps r, d: $G \to U$, where if G is thought of as a category, $r(x)$ is the range of x and $d(x)$ is the domain of x. Thus $d(x) = x^{-1}x$ and $r(x) = x\,x^{-1}$.

1.5 Let λ be a finite measure on G. Let μ be the measure $d^*\lambda$ on U. We have a decomposition $\lambda = \int_U \lambda_u d\mu(u)$ of λ as a direct integral of measures λ_u supported by $d^{-1}(u)$. That is, given a Borel set $A \subseteq G$, the map $u \to \lambda_u(A)$ is Borel and $\lambda(A) = \int \lambda_u(A)\, d\mu(u)$.

DEFINITION 1.6. λ is said to be quasi-symmetric if given a Borel set $A \subseteq G$, A is λ-null if A^{-1} is λ-null.

DEFINITION 1.7. Given $x \in G$, we can define the Borel map ρ_x : $d^{-1}(r(x)) \to d^{-1}\,(d(x))$ by $y \to yx$. λ is said to be quasi-invariant if it is quasi-symmetric and $\rho_x{}^*\lambda_{r(x)} \sim \lambda_{d(x)}$ for all $x \in G$. (We can relax this condition a little. It suffices that $(\rho_x)^*\, \lambda_{r(x)} \sim \lambda_{d(x)}$ for all $x \in r^{-1}$ (some set conull in U w.r.t. μ)).

Definition 1.7 can be replaced by a new definition with r in place of d; the two definitions are equivalent.

These definitions seem somewhat technical; the following example should clarify them.

EXAMPLE 1.8. Let S be a standard Γ-space and μ a finite measure on S. Then $\Gamma \times S$ is an analytic groupoid, thus $G^{(2)} = \{((\beta, \alpha \cdot x), (\alpha, x)):$ $\beta, \alpha \in \Gamma, x \in S\}$ and $(\beta, \alpha \cdot x)\,(\alpha, x) = (\beta\alpha, x)$. So $U = S$. Let λ' be a probability measure on Γ equivalent to Haar measure. We take $\lambda =$

$\lambda' \times \mu$ so $\lambda_u = \lambda'$ for all $u \in U = S$. The quasi-invariance of λ follows from the invariance of Haar measure and the quasi-invariance of μ.

DEFINITION 1.9. A Borel set $T \subseteq U$ is said to be saturated if, for all $x \in G$, $d(x) \in T$ iff $r(x) \in T$.

DEFINITION 1.10. The pair (G, C) consisting of an analytic groupoid G and a measure class, $C = [\lambda]$, is said to be an ergodic groupoid if λ is quasi-invariant (Definition 1.7) and if, for all saturated Borel sets T (Definition 1.9), either $(d^*\lambda)(T) = 0$ or $(d^*\lambda)(U - T) = 0$. G is said to be a proper ergodic groupoid if, whenever T is the saturation of a single point in U, $(d^*\lambda)(T) = 0$.

DEFINITION 1.11. An ergodic equivalence relation is an ergodic groupoid in which, given x and y such that $d(x) = d(y)$ and $r(x) = r(y)$, then $x = y$. If u, $v \in U$ and there is an $x \in G$ such that $u = d(x)$, $v = r(x)$, we say $u \sim v$.

EXAMPLE 1.12. In Example 1.8 we constructed an analytic groupoid $\Gamma \times S$ and a quasi-invariant measure $\lambda' \times \mu$. If the action of Γ on S is ergodic, $(\Gamma \times S, [\text{Haar} \times \mu])$ is an ergodic groupoid.

1.13. *Restriction*

Given a groupoid G and a set $E \subseteq U$, we can define the restriction of G to E, $G \mid E$, as $\{x \in G: d(x) \in E \text{ and } r(x) \in E\}$. If G is an analytic groupoid and E is a Borel set in U, $G \mid E$ is an analytic groupoid. If (G, C) is an ergodic groupoid, let $C = [\lambda]$, $\mu = d^*\lambda$, suppose $\mu(E) > 0$, then $(G \mid E, C \mid E)$ is an ergodic groupoid. If $\mu(E) = 0$, we might be able to find a measure class C' on $G \mid E$ such that $(G \mid E, C')$ is an ergodic groupoid.

All the ergodic groupoids discussed in this thesis will be restrictions of $(\Gamma \times S, [\text{Haar} \times \mu])$ (see example 1.12).

1.14. *Homomorphisms of Ergodic Groupoids*

Since this thesis can be understood by a reader unfamiliar with the technicalities of the definitions of strong and weak homomorphisms, we will not discuss them here. A homomorphism should be thought of as a homomorphism of groupoids, which is a Borel map, and has some suitable property with respect to the measure classes. For the details consult [3].

Definition 1.15. Similarity of homomorphisms. Once again ignoring technicalities, we say two homomorphisms ϕ, ψ: $G_1 \to G_2$ are similar and write $\phi \simeq \psi$ if there is a Borel map θ: $U_1 \to G_2$, where U_1 consists of the units of G_1 and $\theta(r(x))\,\phi(x) = \psi(x)\,\theta(d(x))$, i.e., θ corresponds to a natural transformation of functors.

Definition 1.16. Two ergodic groupoids G_1 and G_2 are said to be similar ($G_1 \simeq G_2$) if there are homomorphisms ϕ: $G_1 \to G_2$ and ψ: $G_2 \to G_1$ such that $\phi \circ \psi \simeq \mathrm{id}$ on G_2 and $\psi \circ \phi \simeq \mathrm{id}$ on G_1.

Definition 1.17. A similarity class of ergodic groupoids is said to be a virtual group.

1.18 A Theorem due to Arlan Ramsay [3, Theorem 6.17, p. 290]:

Given an ergodic groupoid (G, C) and a Borel set E in U whose saturation is conull in U (the saturation of E is $r(d^{-1}(E))$ and so is analytic), there is a measure $\tilde{\lambda}$ on $G \mid E$ such that $(G \mid E, [\tilde{\lambda}])$ is an ergodic groupoid similar to (G, C). If $\lambda \in C$ and $(d^* \lambda)(E) > 0$, we may take $\tilde{\lambda}$ as the restriction of λ.

In Section II we will state and prove a slight modification of this theorem (Proposition 2.13).

1.19 This theorem of Arlan Ramsay (or its slight modification) is all we use in this thesis to prove two ergodic groupoids are similar. However, the same ergodic groupoid can be described in various ways. For example, we may be given an ergodic action of Γ on (S, μ) and so obtain the ergodic groupoid $(\Gamma \times S, [\text{Haar} \times \mu])$ and then restrict to some Borel set F to obtain a similar ergodic groupoid $(\Gamma \times S \mid F, [\text{Counting} \times \tilde{\nu}])$. If the action of Γ is free (i.e., if $\alpha \cdot x = x$, $\alpha = \mathrm{id}$), this is just an ergodic equivalence relation and we can ignore the action of Γ. So it may happen there is an action, say, of $\mathbb{Z}$ on F preserving $\tilde{\nu}$ whose orbits are just the restrictions to F of those of Γ. Then $(\Gamma \times S \mid F, [\text{Counting} \times \tilde{\nu}])$ is the same ergodic groupoid as $(\mathbb{Z} \times F, [\text{Counting} \times \tilde{\nu}])$ which is defined by an ergodic action of $\mathbb{Z}$ on $(F, \tilde{\nu})$. Thus the action of Γ on (S, μ) and the action of $\mathbb{Z}$ on $(F, \tilde{\nu})$ define the same virtual group.

1.20. Let us assume that we had a complete classification of the virtual subgroups of Γ. To obtain a classification of the ergodic actions of Γ, we have to investigate the imbeddings of virtual groups in Γ. For the sake of simplicity, assume a virtual group is presented to us by an ergodic action of a locally compact separable group Γ_2 on (S_2, μ_2). It follows that we have to describe all ergodic Γ-spaces (S, μ) such that $(\Gamma_2 \times S_2, [\text{Haar} \times \mu_2])$ is similar to $(\Gamma \times S, [\text{Haar} \times \mu])$. To this end we have the following construction:

1.21. *The Image of a Homomorphism*

The image of a homomorphism [7] is the appropriate generalization of the flow built up under a function [16].

Let $\phi: \Gamma_2 \times S_2 \to \Gamma$ be a homomorphism of ergodic groupoids. We can define an action of $\Gamma_2 \times \Gamma$ on $(S_2 \times \Gamma, \mu_2 \times \text{Haar})$ by $(\alpha, \beta) \cdot (s, \gamma) = (\alpha \cdot s, \phi(\alpha, s)\gamma\beta^{-1})$. Consider the corresponding Boolean $(\Gamma_2 \times \Gamma)$-space, **A**. Let **B** be the Boolean σ-algebra of Γ_2-invariant elements of **A**, then **B** is a Boolean Γ-space and corresponds to an ergodic Borel Γ-space which is defined to be the image of ϕ.

Suppose there is a Borel cross section E for the Γ_2-orbits in $(S_2 \times \Gamma)$, i.e., a Borel set E such that $\Gamma_2 \cdot E$ is conull in $S_2 \times \Gamma$ but which intersects each Γ_2-orbit in at most one point. Then we have a projection $\pi: S_2 \times \Gamma \to E$ mapping a point to the intersection of its Γ_2-orbit with E. We can define an action $*$ of Γ on E by $\alpha * x = \pi(\alpha \cdot x)$. With this action, $(E, \pi^*(\mu_2 \times \text{Haar}))$ becomes an ergodic Γ-space and is isomorphic to the image of ϕ.

1.22 Every Γ-space (S, μ) defining the same virtual group as the Γ_2-space (S_2, μ_2) is isomorphic to the image of some homomorphism $\varphi: (\Gamma_2 \times S_2, [\text{Haar} \times \mu_2]) \to (\Gamma, [\text{Haar}])$ (see Proposition 6.1). Thus we have some sort of description of all possible imbeddings of the virtual group in Γ.

This description is somewhat inadequate. In the first place, we cannot find, in general, the cross-section E for the Γ_2-orbits (see 1.21). Also, we have no criterion for deciding when two ergodic Γ-spaces are isomorphic.

II. Reduction to the Discrete Case

Let Γ be a locally compact separable group. Let Γ act in a proper free ergodic fashion on $(S, [\mu])$, where S is a complete metric space and μ is a quasi-invariant probability measure. We may assume that S is a Borel subset of the universal Γ-space [1, p. 329]. Let d be the metric on S.

Let Λ be a compact neighborhood of id. Let Δ be a compact neighborhood of id such that $\Delta^2 \subsetneq \Lambda$. Assume $\Lambda = \Lambda^{-1}$. Let $A_\epsilon = \{s \in S$: if $d(s, \alpha \cdot s) \leqslant \epsilon$, either $\alpha \in \operatorname{Int} \Delta$ or $\alpha \notin \Lambda\}$.

PROPOSITION 2.1.

$$\bigcup_n A_{1/n} = S.$$

Proof. If $s \in S - \bigcup_n A_{1/n}$, then for all n there exists an $\alpha_n \in \Lambda - \operatorname{Int} \Delta$ such that $d(s, \alpha_n \cdot s) \leqslant \epsilon$. Λ-Int Δ is compact, therefore there exists a convergent subsequence $\alpha_{n_r} \to \alpha \in \Lambda$-Int Δ; so $d(s, \alpha_n \cdot s) \to d(s, \alpha \cdot s)$, and hence $d(s, \alpha \cdot s) = 0$, therefore $s = \alpha \cdot s$.

However, Γ acts freely on S, therefore $\alpha = \text{id}$. But $\alpha \in \Lambda$-Int Δ. This contradiction proves that $\bigcup_n A_{1/n} = S$.

PROPOSITION 2.2. *A_ϵ is open.*

Proof. If $s_n \to s$ and $s_n \notin A_\epsilon$ for all n, there exists an $\alpha_n \in \Lambda$-Int Δ such that $d(s_n, \alpha_n \cdot s_n) \leqslant \epsilon$.

α_n has a subsequence $\alpha_{n_r} \to \alpha \in \Lambda$-Int Δ, therefore $d(s, \alpha \cdot s) \leqslant \epsilon$, so $s \notin A_\epsilon$. Therefore A_ϵ is open.

PROPOSITION 2.3. *There exists an open ball B, of positive measure and of radius $<\epsilon/2$ contained in A_ϵ, for some $\epsilon > 0$.*

Proof. By Proposition 2.1, there exists an $\epsilon > 0$ such that A_ϵ is of positive measure.

By Proposition 2.2, A_ϵ is covered by a countable family of open balls of radius $<\epsilon/2$ contained in A_ϵ, therefore we can choose an open ball of positive measure.

PROPOSITION 2.4. *If $s \in B$, $B \cap (\operatorname{Int} \Delta) \cdot s = B \cap \Lambda \cdot s$.*

Proof. If $t \in B \cap \Lambda \cdot s$, $t = \alpha \cdot s$ for some $\alpha \in \Lambda$. $d(s, t) < \epsilon$ and $s \in A_\epsilon$, therefore $\alpha \in \operatorname{Int} \Delta$ or $\alpha \notin \Lambda$. Therefore $\alpha \in \operatorname{Int} \Delta$, so $t \in B \cap (\operatorname{Int} \Delta) \cdot s$.

DEFINITION 2.5. Define an equivalence relation $\sim$ on B by $s_1 \sim s_2$ if there exists an $\alpha \in \operatorname{Int} \Delta$ such that $s_1 = \alpha \cdot s_2$. That this is indeed an equivalence relation follows from Proposition 2.4 and $\Delta = \Delta^{-1}$.

DEFINITION 2.6. If $D \subseteq B$, let $[D] = \{s \in B$: there exists some $t \in D$ such that $s \sim t\}$.

PROPOSITION 2.7. *If D is open in B, so is $[D]$.*

Proof. If D is open, $[D] = (\Lambda \cdot D) \cap B = \cup_{\alpha \in \Lambda} (\alpha \cdot D \cap B)$.

PROPOSITION 2.8. *The topology on $B/\sim$ has a countable basis for open sets.*

Proof. Choose a countable basis $\mathbf{B}$ for B. By Proposition 2.7, $\{[D]/\sim: D \in \mathbf{B}\}$ is a basis for $B/\sim$.

PROPOSITION 2.9. *The topology on $B/\sim$ is T_1.*

Proof. By Proposition 2.4, $[s] = B \cap \Lambda \cdot s$, if $s \in B$, $\Lambda \cdot s$ is compact in S and hence is closed in S. Therefore $[s]$ is closed relative to B, i.e., $B/\sim$ is T_1.

PROPOSITION 2.10. *There exists a Borel set $E \subseteq B$ such that*

(a) *$\Delta \cdot E$ is of positive measure,*

(b) *$m: \Delta \times E \to \Delta \cdot E$, defined by $m(\alpha, x) = \alpha \cdot x$, is $1:1$.*

Proof. From Proposition 2.8, $B/\sim$ is a countably generated Borel space. From Proposition 2.9, $B/\sim$ is separated, so $B/\sim$ is countably separated in the sense of [2, p. 62].

B is an open set in a separable complete metric space, so (B, μ) is a standard measure space. The projection $F: B \to B/\sim$ is continuous. By von Neumann's Cross Section Theorem [2, Theorem 2.1, p. 66], there exists a Borel set $E \subseteq B$ such that F is $1:1$ on E and $\mu(F^{-1}(B/\sim - F(E)) = 0$, i.e., $\mu(B - [E]) = 0$. $F: E \to B/\sim$ is $1:1$ and $\Delta^{-1}\Delta \subseteq \Lambda$ so m is $1:1$. Also, $\Delta \cdot E \supseteq [E]$ and so is of positive measure.

DEFINITION 2.11. Define the measure ν on E by $\nu(A) = \mu(\Delta \cdot A)$. For all Borel sets, $A \subseteq E$. Since $m: \Delta \times E \to S$ is $1:1$, ν is a measure.

PROPOSITION 2.12. *Suppose Δ is closed under conjugation and μ is Γ-invariant. Given a Borel set $A \subseteq E$ and a Borel map $\sigma: A \to \Gamma$, if $s(x) \equiv \sigma(x) \cdot x \in E$, then $\nu(s(A)) \leqslant D\nu(A)$, where for some $\alpha_i \in \Gamma$,*

$$\Delta^2 \subseteq \bigcup_{i=1}^{D} \alpha_i \Delta.$$

Proof. We can partition Γ into countably many Borel sets Γ_n such that $\Gamma_n \subseteq \beta_n \Delta$ for suitable $\beta_n \in \Gamma$. Let $A_n = \sigma^{-1}(\Gamma_n)$. $s(A_n) \subseteq \Gamma_n \cdot A_n$ and so

$$\nu(s(A_n)) \leqslant \mu(\Delta\Gamma_n \cdot A_n) \leqslant \mu(\Delta\beta_n\Delta \cdot A_n) = \mu(\Delta^2 \cdot A_n)$$
$$\leqslant \sum_{i=1}^{D} \mu(\beta_i\Delta \cdot A_n) = D\mu(\Delta \cdot A_n) = D\nu(A_n).$$

Therefore, $\nu(s(A)) \leqslant D\nu(A)$.

PROPOSITION 2.13. *($\Gamma \times S$, [Haar $\times\ \mu$]) is similar* (in the sense of [3, Definition 4.9, p. 279] *to*

$$(\Gamma \times S \mid E, [\text{Counting} \times \nu]).$$

LEMMA.

$$(\Gamma \times S \mid E, [\text{Counting} \times \nu])$$

is an ergodic groupoid.

Proof. [Counting $\times\ \nu$] is obviously right-invariant (in the sense of [3, p. 274]). A Borel set A in $(\Gamma \times S \mid E)$ is null iff $d(A)$ is null. Therefore A^{-1} is null iff $r(A)$ is null. We claim $d(A)$ is null iff $[d(A)]$ is null and so [Counting $x\nu$] is symmetric. Given A, a Borel set in E, $\nu(A) = \mu(\Delta \cdot A)$, Γ is covered by countably many translates of Δ, therefore $\nu(A) = 0$ iff $\mu(\Gamma \cdot A) = 0$, i.e., iff $\nu([A]) = 0$. That [Counting $\times\ \nu$] is ergodic is obvious.

Proof. (The method used is due to Arlan Ramsay [3, Theorem 6.17, p. 290]).

From [3, Theorem 6.18, p. 292] and Proposition 1.10(a), we see that $\Gamma \times S$ is similar to its restriction to $\Delta \cdot E$. We define $\phi\colon \Gamma \times S \mid E \to \Gamma \times S \mid \Delta \cdot E$ to be the inclusion. We define $\psi\colon \Gamma \times S \mid \Delta \cdot E \to \Gamma \times S \mid E$ by $\psi(\alpha, \beta \cdot x) = (\gamma\alpha\beta, x)$, where γ is the unique element of Δ such that $\gamma\alpha\beta \cdot x \in E$.

It is trivial algebra to see that ϕ and ψ are algebraically homomorphisms. ϕ is obviously a strict homomorphism (in the sense of [3, Definition 6.1, p. 286]. Let $\tilde{\psi} = \psi \mid$ units. Then $\tilde{\psi}(\beta \cdot x) = x$ for all $\beta \in \Delta$ and for all $x \in E$. Therefore, if F is a Borel set in E, $\tilde{\psi}^{-1}(F) = \Delta \cdot F$.

Therefore $\nu(F) = \mu(\psi^{-1}(F))$, and so in particular if the saturation of F is null, so is F and therefore $\tilde{\psi}^{-1}(F)$ is null. Thus from Definition 6.1

[3, p. 286] we see that ψ is a strict homomorphism. $\psi \circ \phi$ is the identity on $\Gamma \times S \mid E$ and $\phi \circ \psi = \psi$.

Let $\theta(\beta \cdot x) = \beta, x)$ for all $\beta \in \Delta$, $x \in E$. Then $\theta(\alpha\beta \cdot x)\, \psi(\alpha, \beta \cdot x) = (\gamma^{-1}, \gamma\alpha\beta \cdot x)(\gamma\alpha\beta, x) = (\alpha\beta, x) = (\alpha, \beta \cdot x)(\beta \cdot x) = (\alpha, \beta \cdot x)\, \theta(\beta \cdot x)$. So $\phi \circ \psi$ is strictly similar to the identity (in the sense of [3, Definition 6.4, p. 286]). Thus $(\Gamma \times S$, [Haar $\times \mu$]) is similar to $(\Gamma \times S \mid E$, [Counting $\times \nu$]).

Proposition 2.14. *If Γ acts in a proper ergodic fashion on $(S, [\mu])$, ν is atom-free.*

Proof. If x is an atom of ν, $\mu(\Delta \cdot x) > 0$, so $\Gamma \cdot x$ is conull in S.

Remark 2.15. Since the measure on $(\Gamma \times S \mid E)$ is Counting $\times\, \nu$, if A is Borel in $(\Gamma \times S \mid E)$, A is null iff $d(A)$ is null iff $r(A)$ is null. It follows that a Borel set in E is null iff its saturation is null.

These facts will often be used in conjuction with von Neumann's Cross Section Theorem to obtain cross-sections for the map $d: A \to E$, defined by $(\alpha, x) \to x$.

We are now able to prove Warren Ambrose's Theorem on the representation of ergodic actions of $\mathbb{R}$ as "flows built up under a function" [6].

Proposition 2.16.[1] *Any nontrivial ergodic action of $\mathbb{R}$ on (S, μ) is isomorphic to the "flow built up under a function" from an ergodic action of $\mathbb{Z}$ on some (E, ν). If μ is $\mathbb{R}$-invariant and finite, we may assume ν is $\mathbb{Z}$-invariant.*

Proof. If the action of $\mathbb{R}$ is to be proper ergodic, it must be free (because a compact group has no proper ergodic actions. Also see Proposition 7.1). The transitive case is trivial ($\mathbb{Z}$ acts trivially on a single-point space). Let $\Delta = [-\frac{1}{2}, +\frac{1}{2}]$ and use Proposition 1.10 and Definition 1.11 to obtain (E, ν).

If H is a Borel set in $\mathbb{R}$, $\{x \in E: (H \cdot x) \cap E = \varnothing\}$ is ν-measurable. It follows that, by removing a null set from E, we may assume $\{x \in E: ((0, \infty) \cdot x) \cap E = \varnothing\}$ and $\{x \in E: ((-\infty, 0) \cdot x) \cap E = \varnothing\}$ are empty, otherwise we obtain a measurable cross-section for the orbits of $\mathbb{R}$ in almost all of S. Again removing a ν-null set in E (and hence a μ-null set in S), we may assume that, for all rationals q, $\{x \in E: [-q, 0) \cdot x \cap E = \varnothing\}$ is Borel. We can then define $f: E \to R$, a Borel map by $f(x) =$

[1] The measure invariant case is due to Warren Ambrose.

inf{rationals q s.t. $[-q, 0) \cdot x \cap E \neq \varnothing\}$. Since $m\colon \Delta \times E \to S$ is $1:1$, $f(x)$ is then the least real number α such that $\alpha \cdot x \in E$ and $\alpha > 0$.

We define the action of $\mathbb{Z}$ on (E, ν) by $1 \cdot x = f(x) \cdot x$. This is a $1:1$ onto Borel map and so is a Borel isomorphism. A is null in $(\Gamma \times S)| E$ iff $r(A)$ is null iff $d(A)$ is null. So if B is null in E, $d^{-1}(B)$ is null in $\Gamma \times S \mid E$ and so $r(d^{-1}(B)) = [B]$ is null. It follows that $1 \cdot B \subseteq [B]$ is null. Similarly, if $1 \cdot B$ is null, $B \subseteq [1 \cdot B]$ is null. Thus the action of $\mathbb{Z}$ preserves the measure class $[\nu]$. If we build up a flow under the function f, we obtain the original action of $\mathbb{R}$ on some conull invariant set in S. Also, we obtain the measure class $[\mu]$ on S.

Suppose μ is $\mathbb{R}$-invariant. Given an integer r, denote the action of r on E by τ_r. By Proposition 2.12, if A is a Borel set in E, $(\tau_r{}^*\nu)(A) \leqslant D\nu(A)$ for some integer D. Let

$$f(x) = \limsup_{n\to\infty} (2n+1)^{-1} \sum_{r=-n}^{+n} \frac{d(\tau_r{}^*\nu)}{d\nu}(x).$$

Then $D^{-1} \leqslant f(x) \leqslant D$ a.e. and $\tilde{\nu}(A) = \int_A f(x)\, d\nu(x)$ defines a $\mathbb{Z}$-invariant measure on E equivalent to ν.

If we build up the flow under f using this $\mathbb{Z}$-invariant measure $\tilde{\nu}$, we obtain an $\mathbb{R}$-invariant measure $\tilde{\mu}$ equivalent to μ. Therefore $d\tilde{\mu}/d\mu$ is an $\mathbb{R}$-invariant Borel function and so is a constant, say, c. Thus if we replace ν by $c^{-1}\tilde{\nu}$, we may assume that ν is $\mathbb{Z}$-invariant and that μ is not just equivalent to the measure defined in the "flow built up under a function" construction, but is identical.

III. The Class of Groups for Which the Theorems Will Hold

Definition 3.1. A locally compact separable group Γ will be said to have "property 3.1" if there exists a compact neighborhood Δ of id such that

(a) Δ generates Γ, i.e., $\Gamma = \bigcup_{n=1}^{\infty} \Delta^n$,

(b) Δ is closed under conjugation, i.e., for all $\alpha \in \Gamma$, $\alpha\Delta\alpha^{-1} \subseteq \Delta$.

Definition 3.2. Γ is said to have "property 3.2" if there exists a Δ as in Definition 3.1 and there exists an integer D such that, for all integers n, Δ^{2n} is covered by D translates of Δ^n. The following proposition shows that condition (b) of Definition 3.1 is not as strong as it seems.

PROPOSITION 3.3. *If Γ is a compactly generated separable group and has bounded conjugacy classes, then Γ has property* 3.1.

Proof. To say Γ has bounded conjugacy classes means that for all $\alpha \in \Gamma$, the conjugacy class $[\alpha] = \{\beta\alpha\beta^{-1}: \beta \in \Gamma\}$ is contained in some compact set. Let Λ be a compact generating set such that $\Lambda = \overline{\operatorname{Int} \cdot \Lambda}$. Then $\Gamma = \bigcup_n \{\alpha: [\alpha] \subseteq \Lambda^n\}$, $\{\alpha: [\alpha] \subseteq \Lambda^n\} = \bigcup_{\beta\in\Gamma} \beta(\Lambda^n)\beta^{-1}$ and so is closed.

By Baire's Category Theorem for separable locally compact regular spaces [4, p. 200], there exists an n such that $\{\alpha: [\alpha] \subseteq \Lambda^n\}$ has a nonempty interior Ω.

There exist $\beta_1, \dots, \beta_m \in \Gamma$ such that $\Lambda \subseteq \bigcup_{i=1}^m \beta_i\Omega$. Therefore

$$\bigcup_{\alpha\in\Lambda} [\alpha] \subseteq \bigcup_{i=1}^{m} [\beta_i]\, \Lambda^n \subseteq \Lambda^N$$

for some N. Let $\Delta = \overline{\bigcup_{\alpha\in\operatorname{Int}\Lambda} [\alpha]}$. Then $\Lambda^N \supseteq \Delta \supseteq \Lambda$, so Δ is a compact neighborhood of id generating Γ.

Also, Δ is closed under conjugation so Γ has property 3.1. The next two propositions describe some groups with properties 3.1 and 3.2.

PROPOSITION 3.4. *An extension of a compact separable group by a group with properties* 3.1 *and* 3.2 *also has properties* 3.1 *and* 3.2.

Proof. We have $\tau: \Gamma \to \Gamma_1$ where $\operatorname{Ker}\tau$ is compact. Γ and Γ_1 are separable locally compact groups and Γ_1 has properties 3.1 and 3.2. We choose $\Delta_1 \subseteq \Gamma_1$ as in Definitions 3.1 and 3.2. Let $\Delta = \tau^{-1}(\Delta_1)$. It is immediate that Δ is compact neighborhood generating Γ and that Δ is closed under conjugation. So Γ has property 3.1.

There exists an integer D_1 such that for all n, Δ_1^{2n} is covered by D_1 translates of $\Delta_1{}^n$.

Δ_1 is closed under conjugation, so Δ_1^{4n} is covered by $D_1{}^2$ translates of $\Delta_1{}^n$. Let $D = D_1{}^2$. We have $\alpha_1, \dots, \alpha_D \in \Gamma$ such that $\Delta_1^{4n} \subseteq \bigcup_{i=1}^D \tau(\alpha_i)\Delta_1{}^n$. Therefore $\Delta^{4n} \subseteq \bigcup_{i=1}^D \alpha_i\Delta^n$ $(\operatorname{Ker}\tau) \subseteq \bigcup_{i=1}^D \alpha_i\Delta^{2n}$ if $n \geqslant 1$. Therefore $\Delta^{4n+2} \subseteq \bigcup_{i=1}^D \alpha_i\Delta^{2n+1}$ if $n \geqslant 2$. Thus Γ has property 3.2.

PROPOSITION 3.5. $\mathbb{R}^m \times \mathbb{Z}^k$ *has properties* 3.1 *and* 3.2.

Proof. Let $\Delta = \{(a_1, \dots, a_{m+k}) \in \mathbb{R}^m \times \mathbb{Z}^k$ s.t. $-1 \leqslant a_i \leqslant +1$, $i = 1, \dots, m+k\}$. Property 3.1 is obvious. Also $\Delta^n = \{(a_1, \dots, a_{m+k})$ s.t.

$-n \leqslant a_i \leqslant n\}$. Therefore $\Delta^{2n} \subseteq \cup \{(a_1, \dots, a_{m+k})\Delta^n$ s.t. $a_i = -n$, 0, or $+n\}$. Therefore we may take $D = 3^{m+k}$ and $\Gamma = \mathbb{R}^m \times \mathbb{Z}^k$ has properties 3.1 and 3.2.

IV.

In Section IV we assume Γ is a locally compact separable group with properties 3.1 and 3.2. Γ acts on S as in Section II and E has been chosen as in Proposition 2.10, ν as in Definition 2.11. So we have the ergodic groupoid $((\Gamma \times S)| E$, [Counting $\times \nu$]). Γ acts freely, so this ergodic groupoid is an ergodic equivalence relation on E. In this section we show how to describe this equivalence relation in a new, and more useful, way.

From 4.1 to 4.6 we discuss the equivalence relation associated with a directed system of idempotents. This will be the new way of describing the ergodic groupoid.

Definition 4.1. A map $f: E \to E$ is said to be an idempotent if:

(a) f is a Borel map.

(b) $\nu(\operatorname{Dom} f) > 0$. Note that $\operatorname{Dom} f$ need not be conull in E.

(c) $f^*(\nu \mid \operatorname{Dom} f) \sim (\nu \mid \operatorname{Ran} f)$.

(d) $\operatorname{Ran} f \subseteq \operatorname{Dom} f$ and $f \mid \operatorname{Ran} f = \operatorname{id} \mid \operatorname{Ran} f$.

Definition 4.2. A Borel map $f: E \to E$ is said to be liftable if there is a Borel map $\phi: E \to \Gamma$ such that $f(x) = \phi(x) \cdot x$, for all $x \in \operatorname{Dom} f$.

Definition 4.3. Suppose Λ is a neighborhood of id in Γ. An idempotent $f: E \to E$ is said to have domain admitting a border of width Λ if, given $x \in \operatorname{Dom} f$ and $\alpha \in \Lambda$ such that $\alpha \cdot x \in \operatorname{Dom} f$, then $f(x) = f(\alpha \cdot x)$. Thus the sets $f^{-1}(y)$ are at least "Λ apart."

Definition 4.4. $\{f_{m,n} : 0 \leqslant m < n\}$ is said to be a directed system of idempotents if:

(a) each $f_{m,n}$ is an idempotent (Definition 4.1),

(b) if $k < m < n$, $\operatorname{Dom} f_{m,n} = \operatorname{Ran} f_{k,m}$ and $f_{m,n} \circ f_{k,m} = f_{k,n}$.

Note that $\operatorname{Dom} f_{m,n}$ does not depend on n, call it F_m.

DEFINITION 4.5. Given a directed system of idempotents $\{f_{m,n}\}$, define the "associated equivalence relation" on F_0 thus: $x \sim y$ if, for some n, $f_{0,n}(x) = f_{0,n}(y)$. Obviously this is a Borel equivalence relation.

In Section IV we will find a directed system of idempotents such that the associated equivalence relation on F_0 is the same as that defined by the action of Γ.

PROPOSITION 4.6. The Disjoint Graph Lemma. *Let G be a proper ergodic equivalence relation (Definition 1.11). Let $\phi_1, \ldots, \phi_r$ be Borel liftings of $d\colon G \to U$ (see 1.4), i.e., $d \circ \phi = \mathrm{id} \mid \mathrm{Dom}\, \phi$. Let $f_i(u) = r(\phi_i(u))$. If*

(a) $D = \bigcap_{i=1}^{r} \mathrm{Dom}\, \phi_i$ *is of positive measure in U, and*

(b) $\phi_i(u) \neq \phi_j(u)$, *unless $i = j$,*

then there is a Borel set $X \subseteq D$ of positive measure such that the sets $f_i(X)$, $i = 1, \ldots, r$, are disjoint.

Suppose G is the restriction to E of $\Gamma \times S$, then ϕ_i corresponds to a map $\sigma_i : E \to \Gamma$ where $\phi_i(u) = (\sigma_i(u), u)$ and so the range of ϕ_i is the graph of σ_i. Hence the condition that if $i \neq j$, $\phi_i(u) \neq \phi_j(u)$, can be interpreted as saying that the graphs of the σ_i are disjoint.

Proof. G is a proper ergodic groupoid, therefore $\mu = d^*\lambda$ is atom-free. We choose $\lambda \in C$ so that $\mu(U) = 1$. U is separable and λ is atom-free, so we can identify (U, λ) with ([0, 1], Lebesgue) [5, Theorem C, p. 173].

If $i \neq j$, $\phi_i(u) \neq \phi_j(u)$, therefore $r(\phi_i(u)) \neq r(\phi_j(u)$, i.e., if $i \neq j$, $\{u \in D\colon f_i(u) = f_j(u)\} = \varnothing$. Reindex the ϕ_i, if necessary, so that $Y = \{u \in D\colon f_1(u) < \cdots < f_r(u)\}$ is of positive measure.

If $f_1(u) < \cdots < f_r(u)$, there are rationals $q_1 \cdots q_{r-1}$ such that $f_1(u) < q_1 < f_2(u) < \cdots < q_{r-1} < f_r(u)$. The set of u for which this relation holds is Borel and there are only countably many $(r-1)$-tuples of rationals. Therefore there are rational $q_1 \cdots q_{r-1}$ for which $X = \{u \in D\colon f_1(u) < q_1 < f_2(u) < \cdots < q_{r-1} < f_r(u)\}$ is of positive measure. Obviously the sets $\mathrm{f}_i(X)$ are disjoint.

PROPOSITION 4.7. *Given a compact neigborhood Λ of* id *in Γ and a Borel set $X \subseteq E$ of positive measure, there exists a Borel set $Y \subseteq X$ of positive measure with the property: given $x, y \in Y$ and $\alpha, \beta \in \Lambda$ such that $\alpha \cdot x = \beta \cdot y$, then $x = y$.*

Proof. We partition $\Lambda^{-1}\Lambda$ into a finite number of Borel sets $H_1, \ldots, H_k$

each contained in some translate of Δ. Given $x \in X$, $H_i x \cap X$ contains at most one point. $\{x: (H_i \cdot x) \cap X \neq \varnothing\}$ is Borel and so we can find a Borel set Z of positive measure in X such that on Z $\{i: (H_i \cdot x) \cap X \neq \varnothing\}$ is constant. Relabel the H_i, if necessary, so that $(H_i \cdot x) \cap X \neq \varnothing$ if $i = 1,..., r$ and $(H_i \cdot x) \cap X = \varnothing$ if $i = r + 1,..., k$. Let G be the ergodic equivalence relation $(\Gamma \times S \mid X$, [Counting $\times$ $\nu \mid X])$. Define $\phi_i : Z \to G$ by $\phi_i(x) \in G \cap (H_i \times \{x\})$, i.e., $\phi_i(x)$ is the unique member of H_i such that $\phi_i(x) \cdot x \in X$. Since $d \mid (G \cap (H_i \times Z)) \to Z$ is $1 : 1$ onto and d is Borel, d^{-1} and ϕ_i are also Borel. We may assume that $\phi_1(x) = (\mathrm{id}, x)$.

By the Disjoint Graph Lemma (Proposition 4.6), there is a Borel set $Y \subseteq Z$ of positive measure such that the $f_i(Y)$ are disjoint. It follows that if $\alpha, \beta \in \Lambda$, $x, y \in Y$, and $\alpha \cdot x = \beta \cdot y \in X$, then $x = \alpha^{-1}B \cdot y$ and $\alpha^{-1}\beta \in H_i$ for some i. Therefore $x = f_i(y) \in X \cap f_i(X)$. Therefore $f_i = \mathrm{id}$ and so $x = y$.

Proposition 4.8. *Given a compact symmetric neighborhood Λ of* id *in Γ and a Borel set $X \subseteq E$ of positive measure, there is a Borel set $Y \subseteq X$ such that*:

(a) *given $x, y \in Y$ and $\alpha, \beta \in \Lambda$ such that $\alpha \cdot x = \beta \cdot y$, then $x = y$,*

(b) $(\Lambda^2 \cdot Y) \cap X = X$ (mod 0).

Proof. A Borel set $Y \subseteq E$ is said to have "property P" if, given $x, y \in Y$ and $\alpha, \beta \in \Lambda$ such that $\alpha \cdot x = \beta \cdot y$, then $x = y$. Choose $Y_1 \subseteq X$ a Borel set with property P such that $\nu(Y_1) \geqslant \frac{1}{2} \sup \{\nu(Y)$: Y is a Borel set in X with property $P\}$. If Y_n has been chosen, choose $Y_{n+1} \subseteq X - \Lambda^2 \cdot (Y_1 \cup \cdots \cup Y_n)$ such that $\nu(Y_{n+1}) \geqslant \frac{1}{2} \sup\{\nu(Y)$: Y is a Borel in $X - \Lambda^2 \cdot (Y_1 \cup \cdots \cup Y_n)$ with property $P\}$. Let $Y = \bigcup_n Y_n$. Y obviously has property P. If $\nu(X - \Lambda^2 \cdot Y) > 0$, we apply Proposition 4.7 with $X - \Lambda^2 \cdot Y$ in place of X and obtain a Borel set Y_∞ with property P disjoint from $\Lambda^2 \cdot Y$ and of positive measure. Therefore $\frac{1}{2}\nu(Y_\infty) \leqslant \nu(Y_n)$ for all n, so $\infty > \nu(X) > \sum_{n=1}^{\infty} \nu(Y_n) = \infty$. This contradiction proves that $\nu(X - \Lambda^2 \cdot Y) = 0$, as required.

Proposition 4.9. *Given an integer N and $\delta > 0$, there is an integer M such that, for all Borel sets $X \subseteq E$ of positive measure, there is a liftable idempotent f: $E \to E$ such that*

(a) $\mathrm{Dom}\, f \subseteq X$,

(b) *f has domain admitting a border of width Δ^N* (Definition 4.3),

(c) *if D is the integer in Definition* 3.2 *and* $\epsilon_0 = D^{-3}$, $\nu(\operatorname{Dom} f) \geqslant \epsilon_0 \, \nu(x)$,

(d) $\nu(x \cap (\Delta^N \cdot \operatorname{Dom} f) \leqslant (1 + \delta)\, \nu(\operatorname{Dom} f)$,

(e) *if* $f(x) = f(y)$, $x \in \Delta^M \cdot y$.

Proof. Choose an integer $J > (\delta\epsilon_0)^{-1}$. Let $L = (J + 1)N$. Let $\Lambda = \Delta^{4L}$. Let $M = 8L$. Let Y be chosen as in Proposition 4.8 to have properties 4.8(a) and 4.8(b).

$\Lambda^2 = \Delta^{8L}$ and so is covered by D^3 translates of Δ^L (see Definition 3.2), i.e., $\Delta^{8L} \leqslant \bigcup_{i=1}^{D^3} \Delta^L \alpha_i$ for some suitable $\alpha_i \in \Gamma$.

Therefore $\bigcup_{i=1}^{D^3} ((\Delta^L \alpha_i \cdot Y) \cap X) = X (\operatorname{mod} 0)$.

Therefore there exists an integer i such that $\nu((\Delta^L \alpha_i \cdot Y) \cap X) \geqslant D^{-3} \nu(X) = \epsilon_0 \nu(X)$. By reindexing the α_i, if necessary, we may assume that $\nu((\Delta^L \alpha_1 \cdot Y) \cap X) \leqslant \epsilon_0 \nu(X)$.

In Fig. 1 we consider part of one orbit of $\Gamma = R^2$. The points of

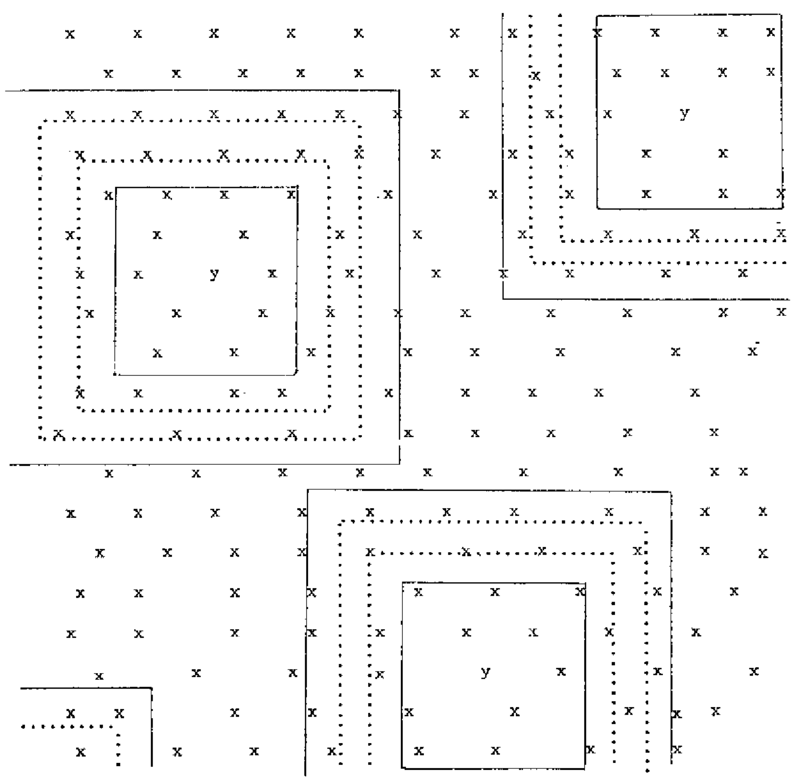

FIGURE 1

$X - Y$ are denoted by x, the points of Y by y. Δ is the square of unit side. The smallest squares in the figure are of side L, the largest of side $2L$. We have just shown that by translating the square by α_1, the points of X in all the smallest squares in all the orbits form a Borel set of measure at least $\epsilon_0 \nu(X)$.

The sets $(\Delta^{L+(r+1)N}\alpha_1 \cdot Y - \Delta^{L+rN}\alpha_1 \cdot Y) \cap X$, $r = 0,\ldots, (J-1)$, are disjoint Borel sets. They correspond in the figure to the translates by α_1 of the regions between the dotted lines. (In the figure, $J = 3$).

It follows that for some r,

$$\nu((\Delta^{L+(r+1)N}\alpha_1 \cdot Y - \Delta^{L+rN}\alpha_1 \cdot Y) \cap X) \leqslant J^{-1}\nu(X) \leqslant \delta\epsilon_0\nu(X).$$

With N and r chosen as above, we define a map $g\colon E \to E$ as follows: $\operatorname{Dom} g = (\Delta^{L+rN}\alpha_1 \cdot Y) \cap X$. $g(\alpha\alpha_1 \cdot y) = y$ for $\alpha \in \Delta^{L+rN}$. This is well-defined since if $\alpha\alpha_1 \cdot y = \beta\alpha_1 \cdot x$ for $y, x \in Y$ and $\alpha, \beta \in \Delta^{L+rN}$, then $\alpha_1^{-1}\beta\alpha\alpha_1 \in \Delta^{2(L+rN)} \subseteq \Lambda$ and $x \in \Lambda \cdot y$ so $x = y$ (Proposition 4.8, property (a)).

Let ψ: $\operatorname{Dom} g$: $\to \Gamma$ be the unique map such that $g(x) = \psi(x) \cdot x$, i.e., $\psi(\alpha\alpha_1 \cdot y) = \alpha_1^{-1}\alpha^{-1}$ if $\alpha \in \Delta^{L+rN}$. If H is a Borel set in Γ, $\psi^{-1}(H) = (H^{-1} \cdot Y) \cap X$ and so is ν-measurable.

Thus ψ is a ν-measurable map: $\operatorname{Dom} g \to \Gamma$. Therefore, if we restrict the domain of g by removing a null set, we may assume ψ is a Borel map. Since g is liftable (to ψ) $g^*(\nu \mid \operatorname{Dom} g) \sim (\nu \mid \operatorname{Ran} g)$ (because a Borel set in E is null iff its saturation is null). By von Neumann's Cross Section Theorem [2, Theorem Z.1, p. 66], there is a Borel set $Z \subseteq \operatorname{Dom} g$ such that $g \mid Z$ is $1:1$ and $g(Z)$ is conull and Borel in $\operatorname{Ran} g$.

Restrict the domain of g to $g^{-1}(g(Z))$, a Borel set conull in $\operatorname{Dom} g$.

Define $f = (g \mid Z)^{-1} \circ g$. It is easily seen that the composite of liftable maps is liftable, and the inverse of $1:1$ liftable map is liftable. Since f is liftable, $f^*(\nu \mid \operatorname{Dom} f) \sim (\nu \mid \operatorname{Ran} f)$. It is easily seen that f is an idempotent (Definition 4.1). $\operatorname{Dom} f \subseteq (\Delta^{L+rN}\alpha_1 \cdot Y) \cap X$ and is conull to $(\Delta^{L+rN}\alpha_1 \cdot Y) \cap X$.

(a) is obious.

(b) follows from the choice of Y as in Proposition 4.8 and $r \leqslant (J-1)$.

(c) follows from $\nu((\Delta^{L}\alpha_1 \cdot Y) \cap X) \geqslant \epsilon_0\nu(x)$.

(d) follows from the choice of r.

(e) $x \in (\Delta^{L+rN}\alpha_1)^{-1}(\Delta^{L+rN}\alpha_1) \cdot f(x) \subseteq \alpha_1^{-1}\Delta^{4L}\alpha_1 \cdot f(x) = \Delta^{4L} \cdot f(x)$. So if $f(x) = f(y)$, $x \in \Delta^{8L} \cdot y = \Delta^{M} \cdot y$.

In Fig. 1 we translate the $(r+1)$st smallest squares by α_1. f maps each point in X of one of these squares to the unique point in Z of the same square. Of course, if $\alpha_1 = \mathrm{id}$, we could take $Z = Y$ and then f just maps each point x in the $(r+1)$st smallest square to the point y in that square.

Remark. The existence of a specific $\epsilon_0 > 0$ rather than an arbitrary $\epsilon > 0$ is convenient but not crucial. However, we have restricted our attention to groups with property 3.2 because we need Proposition 4.9(d).

PROPOSITION 4.10. *Given an integer K and $\epsilon > 0$, there is an integer M and a liftable idempotent $f: E \to E$ such that*:

(a) *the domain of f admits a border of width Δ^K (see Definition* 4.3),

(b) $\nu(\mathrm{Dom}\, f) \geqslant (1-\epsilon)\, \nu(E)$,

(c) *if $f(x) = f(y)$, $x \in \Delta^M \cdot y$.*

Proof. Choose δ so that $(1/(1+\delta)) = 1 - \epsilon$. We define a sequence $\{f_i\}$ of idempotents as follows:

Let $N = 2K$ and apply Proposition 4.9 with $E = X$; we obtain f_1 in place of f. Suppose $f_1, \ldots, f_r$ have been defined, then we let

$$X_{r+1} = E - \bigcup_{s=1}^{r} (X_s \cap (\Delta^N \cdot \mathrm{Dom}\, f_s)).$$

Apply Proposition 4.9 with X_{r+1} in place of X to obtain f_{r+1} in place of f.

The sets $\mathrm{Dom}\, f_r$ are disjoint and so we can define f by $f \mid \mathrm{Dom}\, f_r = f_r$. Since each f_r is a liftable idempotent, f is a liftable idempotent. We now check that f has properties (a), (b), and (c).

(a) Given $x, y \in \mathrm{Dom}\, f$ and $\alpha \cdot x = \beta \cdot y$, where $\alpha, \beta \in \Delta^K$, then $x \in \mathrm{Dom}\, f_r$, $y \in \mathrm{Dom}\, f_s$ for some r, s. We may assume $r \geqslant s$. If $r > s$, $x \in X_r \subseteq X_s$ and $y \in \mathrm{Dom}\, f_s$, so $x \in X_s \cap (\Delta^N \cdot \mathrm{Dom}\, f_s)$, therefore $x \notin X_{s+1}$ and, since $r > s$, $x \notin X_r$, a contradiction. If $r = s$, f_s has domain admitting a boundary of width Δ^N, so $x = y$. Thus f has domain admitting a boundary of width Δ^K.

(b) $\nu(\mathrm{Dom}\, f) = \sum_{r=1}^{\infty} \nu(\mathrm{Dom}\, f_r)$. By Proposition 4.9(c),

$$\nu(\mathrm{Dom}\, f_r) \geqslant \epsilon_0 \nu(X_r) \geqslant \epsilon_0 \left[\nu(E) - \sum_{s=1}^{r} \nu(X_s \cap (\Delta^N \cdot \mathrm{Dom}\, f_s))\right].$$

By Proposition 4.9(d), $\nu(X_s \cap (\Delta^N \cdot \mathrm{Dom}\, f_s)) \leqslant (1+\delta)\, \nu(\mathrm{Dom}\, f_s)$.

Therefore $\nu(\mathrm{Dom} f_r) \geqslant \epsilon_0(\nu(E) - \sum_{s=1}^{r'} (1 + \delta)\, \nu(\mathrm{Dom} f_s)) \geqslant \epsilon_0(\nu(E) - (1 + \delta)\, \nu(\mathrm{Dom} f))$. Therefore, if for some $\eta > 0$, $\nu(\mathrm{Dom} f) \leqslant (1 - \eta)/(1 + \delta)\, \nu(E)$, $\nu(\mathrm{Dom} f_r) \geqslant \epsilon_0\, \eta\nu(E)$ for all r and so $\nu(\mathrm{Dom} f) = \infty$, a contradiction. This shows that $\nu(\mathrm{Dom} f) \geqslant \nu(E)/(1 + \delta) = (1 - \epsilon)\, \nu(E)$.

(c) Since the $\mathrm{Dom} f_s$ are disjoint and $\mathrm{Ran} f_s \subseteq \mathrm{Dom} f_s$, if $f(x) = f(y)$, then x and $y \in \mathrm{Dom} f_s$ for some s, and $f_s(x) = f_s(y)$ and so $x \in \Delta^M \cdot y$.

4.11. Proposition 4.10 is nearly the basic tool we need in Section IV. However, we require that the liftable idempotent f, in addition to having the properties (a), (b), and (c) of Proposition 4.10, is also to be $2^P : 1$. For some integer P.

We first obtain a "nearly $2^P : 1$" idempotent and then remove part of its domain to obtain a $2^P : 1$ idempotent. This "nearly $2^P : 1$" idempotent h is defined as $g \circ f$, where f is as in Proposition 4.10 and g is a new liftable idempotent such that $\mathrm{Dom}\, g = \mathrm{Ran} f$ and such that on $\{x \in \mathrm{Ran} f\colon \# f^{-1}(x) = k\}$, g is $m : 1$ where km is "near" 2^P.

PROPOSITION 4.12. *$(\Gamma \cdot x) \cap E$ is infinite for almost all $x \in E$.*

Proof. If this were not true, using von Neumann's Cross Section Theorem [2, Theorem Z.1, p. 66] for the map $d\colon (\Gamma \times S \mid E) \to E$ ($d(\alpha, x) = x$) and the Disjoint Graph Lemma (Proposition 4.6), we obtain a Borel set $Y \subseteq E$ of positive measure meeting each Γ-orbit at most once. However, $(\Gamma \times S \mid E$, [Counting $\times\, \nu$]) is a proper ergodic groupoid and so the existence of such a set Y is impossible. This contradiction proves Proposition 4.12.

PROPOSITION 4.13. *Given an integer $m > 0$, there is an $m : 1$ liftable idempotent $g : E \to E$.*

Note. By Definition 4.1, $\mathrm{Dom}\, g$ must be of positive measure.

Proof. By Proposition 4.12, $(\Gamma \cdot x) \cap E$ is infinite for almost all $x \in E$. We can apply von Neumann's Cross Section Theorem m times to the map d: $(\Gamma \times S \mid E \to E$, defined by $d(\alpha, x) = x$, to obtain m disjoint Borel sets $G_1, \ldots, G_m \subseteq (\Gamma \times S \mid E)$ such that $d(G_i)$ is conull in E and $d \mid G_i$ is $1 : 1$. We may assume that $G_1 = E$. Let $\psi_i = (d \mid G_i)^{-1}$. Let $(ps)_i(x) = r(\psi_i(x))$, where $r(\alpha, x) = \alpha \cdot x$. By the Disjoint Graph Lemma, there is a Borel set $X \subseteq E$ of positive measure in E such that the

$(ps)_i(X)$ are disjoint. Let $\operatorname{Dom} g = \bigcup_{i=1}^m (ps)_i(X)$, and let $g((ps)_i(x)) = x$, for all $x \in X$. $(ps)_i$ is liftable to ψ_i, a Borel map, and $g \mid (ps_i(X) = ((ps)_i)^{-1}$, so g is liftable. It follows that $g^*(\nu \mid \operatorname{Dom} g) \sim (\nu \mid \operatorname{Ran} g)$. (A Borel set in E is null iff its saturation is null).

$\operatorname{Dom} g \subseteq X$ and so is of positive measure. Also, $\operatorname{Ran} g = X \subseteq \operatorname{Dom} g$ and $g \mid X = ((ps_1)^{-1} = \mathrm{id} \mid X$, so g is a liftable idempotent and is $m : 1$.

PROPOSITION 4.14. *Given an integer $m > 0$, there is a liftable idempotent $g: E \to E$ which is $m : 1$ and whose domain is conull in E.*

Proof. There are liftable $m : 1$ idempotents g_n with disjoint domains such that if h is any liftable $m : 1$ idempotent whose domain is disjoint from $\bigcup_{r=1}^n \operatorname{Dom} g_r$, then $\nu(\operatorname{Dom} h) < 2\nu(\operatorname{Dom} g_{n+1})$. Let $\operatorname{Dom} g = \bigcup_n \operatorname{Dom} g_n$. Let $g \mid \operatorname{Dom} g_n = g_n$. g is a liftable $m : 1$ idempotent. If $\operatorname{Dom} g$ is not conull in E, Proposition 4.13 applied to $(\Gamma \times S) | (E - \operatorname{Dom} g)$ provides an $m : 1$ liftable idempotent h whose domain is disjoint from $\operatorname{Dom} g$. Therefore $\nu(\operatorname{Dom} h) < 2\nu(\operatorname{Dom} g_{n+1})$ for all n. $\nu(E) < \infty$, so we have a contradiction.

PROPOSITION 4.15. *Given an integer $m > 0$ and $\epsilon > 0$, there is an integer M and a liftable idempotent $g: E \to E$ such that:*

(a) *g is $m : 1$,*

(b) *$\nu(E\text{-}\operatorname{Dom} g) < \epsilon$,*

(c) *if $g(x) = g(y)$, $x \in \Delta^M \cdot y$.*

Proof. From Proposition 4.14, we have an $m : 1$ liftable idempotent $g: E \to E$ of conull domain. Let $A_n = \{x: g^{-1}(g(x)) \leqslant \Delta^n \cdot x\}$. If $n > m$, $A_n \subseteq A_m$, and $\bigcup_n A_n = \operatorname{Dom} g$. It follows that for some M, $\nu(E - A_M) < \epsilon$. Proposition 4.15 now follows if we restrict g to A_M.

PROPOSITION 4.16. *Given $\epsilon > 0$, $\delta > 0$, and an integer K, there exists a liftable idempotent $h: E \to E$ and integers T, P such that:*

(a) *h has domain admitting a border of width Δ^K (Definition 4.3),*

(b) *$\nu(\operatorname{Dom} h) \geqslant (1 - \epsilon)\, \nu(E)$,*

(c) *if $h(x) = h(y)$, $x \in \Delta^T \cdot y$,*

(d) *if $y \in \operatorname{Ran} h$, $2^P \leqslant \# h^{-1}(y) \leqslant 2^P(1 + \delta)$.*

Proof. A liftable idempotent f and an integer M are chosen as in Proposition 4.10 (replace ϵ by $\epsilon/2$).

Consider the map $\zeta: y \mapsto \#(f^{-1}(y))$, where $\operatorname{Dom} \zeta = \operatorname{Ran} f$. By applying von Neumann's Cross Section Theorem to the Borel map f, we see that ζ, suitably restricted to a set conull in $\operatorname{Ran} f$, is Borel. So we obtain Borel sets $E_k \subseteq \zeta^{-1}(k)$ such that $\bigcup_k E_k$ is conull in $\operatorname{Ran} f$. Since Δ^M can be covered by finitely many translates of Δ, ζ is bounded by L, say. Thus $\bigcup_{k=1}^{L} E_k$ is conull in $\operatorname{Ran} f$.

There is an integer P and integers m_k such that $2^P \leqslant km_k \leqslant 2^P(1 + \delta)$, for $k = 1,..., L$.

The liftable idempotent g_k and the integer M_k are chosen as in Proposition 4.16 with E_k in place of E, $\nu_k = (f^*\nu)|\, E_k$ in place of ν, and $\epsilon/2$ in place of ϵ. Let $T = 2M + \max M_k$.

On $f^{-1}(\operatorname{Dom} g_k)$, h is defined as $g_k \circ f$. h is the composite of liftable maps and so is liftable. We check that h has properties (a), (b), (c), and (d):

(a) If x and $\alpha \cdot x \in \operatorname{Dom} h$ and $\alpha \in \Delta^K$, $f(x) = f(\alpha \cdot x)$ (by Proposition 4.10(a)). Therefore $h(x) = h(\alpha \cdot x)$.

(b) $\operatorname{Dom} h = \bigcup_{k=1}^{L} f^{-1}(\operatorname{Dom} g_k)$. Therefore $\nu(\operatorname{Dom} h) = \sum_{k=1}^{L} \nu_k(\operatorname{Dom} g_k) \geqslant \sum_{k=1}^{L} (1 - \epsilon/2)\, \nu_k(E_k)$ (Proposition 4.15(b)). Therefore

$$\nu(\operatorname{Dom} h) \geqslant (1 - \epsilon/2) \sum_{k=1}^{L} \nu_k(E_k) = (1 - \epsilon/2) \sum_{k=1}^{L} \nu(f^{-1}(E_k))$$
$$= (1 - \epsilon/2)\, \nu(\operatorname{Dom} f) \geqslant (1 - \epsilon/2)(1 - \epsilon/2)\, \nu(E) \geqslant (1 - \epsilon)\, \nu(E).$$

(c) If $h(x) = h(y)$, $f(x)$ and $f(y)$ belong to the same E_k and $g_k(f(x)) = g_k(f(y))$, so $f(x) \in \Delta^{M_k} \cdot f(y)$ (by Proposition 4.15(c). Now $f(f(x)) = f(x)$, so (by Proposition 4.10(c)), $f(x) \in \Delta^M \cdot x$. Similarly, $f(y) \in \Delta^M \cdot y$. Therefore $x \in \Delta^{2M+M_k} \cdot y \subseteq \Delta^T \cdot y$.

(d) $h^{-1}(y) = f^{-1}(g_k^{-1}(y))$, for some k. g_k is $m_k : 1$, each point of $g_k^{-1}(y) \in E_k$, and so, if $z \in g_k^{-1}(y)$, $\#(f^{-1}(y)) = k$.

Therefore, $\# h^{-1}(y) = km_k$ so $2^P \leqslant \# h^{-1}(y) \leqslant 2^P(1 + \delta)$.

PROPOSITION 4.17. *Given $\epsilon > 0$ and an integer K, there is a liftable idempotent $g: E \to E$ and integers T, P such that:*

(a) *g has domain admitting a border of width Δ^K (Definition 4.3),*

(b) $\nu(\operatorname{Dom} g) \geqslant (1 - \epsilon)\, \nu(E)$,

(c) *if $g(x) = g(y)$, $x \in \Delta^T \cdot y$,*

(d) *g is $2^P : 1$.*

Proof. In Proposition 4.16 we replace ϵ by $\epsilon/2$ and choose δ so that $(1-\epsilon) = (1-\epsilon/2)\,(1+\delta)^{-1}$. We obtain a liftable idempotent h and integers T and P. g will be a suitable restriction of h.

By von Neumann's Cross Section Theorem applied repeatedly to $h: E \to E$, we can find disjoint Borel sets $B_k{}^r$, $r = 1,\ldots, k$, $2^P \leqslant k \leqslant 2^P(1+\delta)$, such that $\bigcup_{r,k} B_k{}^r$ is conull in $\operatorname{Dom} h$, $h \mid B_k{}^r$ is $1:1$, and, if $x \in B_k{}^r$, $h^{-1}(h(x))$ intersects each $B_k{}^r$, $r = 1,\ldots, k$, in exactly one point, so $\# h^{-1}(h(x)) = k$. For a fixed k, take the 2^P largest sets $B_k{}^r$ (i.e., the 2^P sets of largest ν) and let C_k be their union.

Let $g = h \mid \bigcup_k C_k$. $\nu(C_k) \geqslant 2^P/2^P(1+\delta)\ \ \nu(\bigcup_{r=1}^k B_k{}^r)$. Therefore $\nu(\operatorname{Dom}) \geqslant 1/(1+\delta)\ \nu(\operatorname{Dom} h) \geqslant (1-\epsilon/2)\,(1+\delta)^{-1}\,\nu(E)$ (by Proposition 4.16(b). Therefore $\nu(\operatorname{Dom} g) \geqslant (1-\epsilon)\,\nu(E)$. So g has property (b). Property (d) is obvious and properties (a) and (c) follow from Proposition 4.16(a) and (c).

PROPOSITION 4.18. *There is a directed system of liftable idempotents* $\{f_{n,m}\}$ *(Definition* 4.4) *such that*:

(a) *Given n, there is an integer P_n such that $f_{n,n+1}$ is 2^{P_n}*: 1, *and*

(b) *if $F_0 = \operatorname{Dom} f_{0,1}$, the associated equivalence relation (Definition* 4.5) *is the same as that defined by* $(\Gamma \times S) \mid F_0$.

Proof. (A) The construction of the directed system $\{f_{m,n}\}$:

We first find liftable idempotents $g_{n,n+1}$ using Proposition 4.17 as follows:

Choose a sequence $\epsilon_n > 0$, $n = 1, 2,\ldots$, such that $\prod_{n=1}^{\infty} (1-\epsilon_n) > 0$. Replace ϵ by ϵ_1 and K by $K_1 = 1$ in Proposition 4.17 to obtain integers T_1, P_1 and an idempotent $g_{0,1}$ in place of T, P, and g, respectively. Let $\nu_1 = g_{0,1}\,\nu$. Let $E_1 = \operatorname{Ran} g_{0,1}$. Let $K_2 = 2T_1 + 2$ and replace ϵ by ϵ_2, K by K_2, E by E_1, and ν by ν_1 in Proposition 4.17 to obtain T_2, P_2, and $g_{1,2}$. Let $g_{0,2} = g_{1,2} \circ g_{0,1}$. Suppose $g_{0,n}$ has been defined. Let $\nu_n = g_{0,n}^*\,\nu$. Let $E_n = \operatorname{Ran} g_{0,n}$. Let $K_n = 2(T_1 + T_2 + \cdots + T_{n-1}) + n$ and replace ϵ by ϵ_n, E by E_n, ν by ν_n, and K by K_n in Proposition 4.17 to obtain T_n, P_n, and $g_{n,n+1}$. Let $g_{0,n+1} = g_{n,n+1} \circ g_{0,n}$. Let $F_0 = \bigcap_n \operatorname{Dom} g_{0,n}$. Let $F_n = g_{0,n}(F_0)$. Let $f_{n,n+1} = g_{n,n+1} \mid F_n$ and define $f_{n,m}$ accordingly.

(B) F_0 is of positive measure.

Proof. $\{\operatorname{Dom} g_{0,n}\}_{n=1}^{\infty}$ is a decreasing sequence of Borel sets, $\nu(\operatorname{Dom} g_{0,n} - \operatorname{Dom} g_{0,n+1}) = (g_{0,n}^*\,\nu)\,(\operatorname{Ran} g_{0,n} - \operatorname{Dom} g_{n,n+1}) \leqslant$

$\epsilon_n(g_{0,n}^* \nu)$ (Ran $g_{0,n}$) by Proposition 4.17 (b). Therefore $\nu(\text{Dom } g_{0,n} - \text{Dom } g_{0,n+1}) \leqslant \epsilon_n \nu(\text{Dom } g_{0,n})$. Therefore $\nu(F_0) \geqslant \prod_{n=1}^{\infty} (1 - \epsilon_n) \nu(E) > 0$

(C) If $x \in F_n$, $(g_{n,n+1})^{-1} (g_{n,n+1}(x)) \subseteq F_n$.

Proof. Suppose $y \in (g_{n,n+1})^{-1} (g_{n,n+1}(x)) \subsetneq \text{Dom } g_{n,n+1} \subseteq E_n$. $E_n = \text{Ran } g_{0,n}$, therefore there are $x_1 \in F_0$ and $y_1 \in E$ such that $x = g_{0,n}(x_1)$, $y = g_{0,n}(y_1)$. $g_{0,n+1}(x_1) = g_{0,n+1}(y_1)$ and $x_1 \in F_0 = \bigcap_{m=1}^{\infty} \text{Dom } g_{0,m}$. Therefore from the definition of $g_{0,m}$, $y_1 \in \bigcap_{m=n+1}^{\infty} \text{Dom } g_{0,m} = \bigcap_{m=1}^{\infty} \text{Dom } g_{0,m} = F_0$. Therefore $y \in F_n$, as required.

(D) $\{f_{n,m}\}$ is a directed system of liftable idempotents.

Proof. By Proposition 4.17, $g_{n,n+1}$ is liftable, so the $f_{n,m}$ are liftable. Next, we check that the $f_{n,n+1}$ are idempotents (Definition 4.1):

(a) $f_{n,n+1}$ is Borel.

(b) $\text{Dom } f_{n,n+1} = F_n$. $g_{0,n}$ is liftable and the saturation of a null set is null. Therefore $(g_{0,n})^* (\nu \mid \text{Dom } g_{0,n}) \sim (\nu \mid \text{Ran } g_{0,n})$ and so from (B) $\nu(\text{Dom } f_{n,n+1}) > 0$.

(c) $(f_{n,n+1})^* (\nu \mid \text{Dom } f_{n,n+1}) \sim (\nu \mid \text{Ran } f_{n,n+1})$, since $f_{n(n+1}$ is liftable and the saturation of a null set is null.

(d) If $y \in \text{Ran } f_{n,n+1}$, $y = f_{n,n+1}(x)$, $x \in \text{Dom } f_{n,n+1} \subsetneq F_n$, so $y = g_{n,n+1}(x)$ and $g_{n,n+1}(y) = y = g_{n,n+1}(x)$. Therefore, by (C) $y \in F_n$. Thus $y \in F_n \cap \text{Ran } g_{n,n+1} \subseteq F_n \cap \text{Dom } g_{n,n+1} = \text{Dom } f_{n,n+1}$. Therefore $\text{Ran } g_{n,n+1} \subseteq \text{Dom } f_{n,n+1}$. Since $g_{n,n+1} \mid \text{Ran } g_{n,n+1} = \text{id} \mid \text{Ran } g_{n,n+1}$, it follows that $f_{n,n+1} \mid \text{Ran } f_{n,n+1} \mid = g_{n,n+1} \text{ Ran } f_{n,n+1} = \text{id} \mid \text{Ran } f_{n,n+1}$.

$\{f_{n,m}\}$ forms a directed system, since if $k < m < n$, $\text{Ran } f_{k,m} = f_{k,m}(F_m) = g_{k,m}(F_m) = F_m = \text{Dom } f_{m,n}$.

(E) $f_{n,n+1}$ is $2^P : 1$ for some integer P.

Proof. $g_{n,n+1}$ is $2^P : 1$ (Proposition 4.17), and by (C), if

$$x \in \text{Dom } f_{n,n+1} = F_n, g_{n,n+1}^{-1}(g_{n,n+1}(x)) \subseteq F_n$$

and so $f_{n,n+1}$ is also $2^P : 1$.

(F) The associated equivalence relation ,$\sim$, is the same as that defined by $\Gamma \times S \mid F_0$.

Proof. Since each $f_{n,n+1}$ is liftable, if $x \sim y$, $x \in \Gamma \cdot y$.

Conversely, if $x, y \in F_0$ and $x \in \Gamma \cdot y$, there is an integer n for which $x \in \Delta^n \cdot y$ (Property 3.1). From Proposition 4.17(c),

$$f_{0,n}(x) \in \Delta^{T_1+T_2+\cdots+T_n} \cdot x.$$

and

$$f_{0,n}(y) \in \Delta^{T_1+T_2+\cdots+T_n} \cdot y.$$

Therefore

$$f_{0,n}(x) \in \Delta^{2(T_1+\cdots+T_n)+n} \cdot f_{0,n}(y),$$

i.e., $f_{0,n}(x) \in \Delta^{K_{n+1}} \cdot f_{0,n}(y)$, $g_{n,n+1}$ has domain admitting a border of width $\Delta^{K_{n+1}}$, so $g_{n,n+1}(f_{0,n}(x)) = g_{n,n+1}(f_{0,n}(y))$, i.e., $f_{0,n+1}(x) = f_{0,n+1}(y)$. Therefore $x \sim y$.

PROPOSITION 4.19. *If f: $E \to E$ is a liftable 2^P: 1 idempotent, there are P liftable* 2 : 1 *idempotents $g_1, \dots, g_p$ such that*:

(a) Dom g_1 *is conull in* Dom f,

(b) Dom g_r = Ran g_r, *for* $r = 1, \dots, P-1$,

(c) Ran g_P *is conull in* Ran f,

(d) $f \mid \operatorname{Dom} g_1 = g_P \circ g_{p-1} \circ \cdots \circ g_2 \circ g_1$.

Proof. We apply von Neumann's Cross Section Theorem $2^P - 1$ times to find disjoint Borel sets $A_1, \dots, A_{2^P}$ such that $f \mid A_r$ is 1 : 1 and $f(A_r)$ is conull in Ran f. Since f is an idempotents, we may asume that $A_1 \subseteq \operatorname{Ran} f$. Also, we may assume that $f(A_r) = A_1$, for all r. So we have liftable 1 : 1 onto Borel maps $h_{r,s} : A_r \to A_s$ defined by $h_{r,s} = (f \mid A_r)^{-1} \circ (f \mid A_s)$.

We define g_k with domain $\bigcup \{A_r : 1 \leqslant r \leqslant 2^{P-k}\}$ thus:

$$\text{if} \quad 1 \leqslant r \leqslant 2^{P-k-1}, \qquad g_k \mid A_r = \text{id}.$$

$$\text{if} \quad 2^{P-k-1} < r \leqslant 2^{P-k}, \qquad g_k \mid A_r = h_{r,s}, \qquad \text{where} \quad s = r - 2^{P-k-1}.$$

The g_r so obtained are liftable idempotents with the required properties.

PROPOSITION 4.20. (A slight modification of Proposition 4.18). *There is a directed system of liftable idempotents $(h_{n,m}\}$ such that*:

(a) $h_{n,n+1}$ *is* 2 : 1,

(b) *if* $H_n = \mathrm{Dom}\, h_{n,n+1}$, *the associated equivalence relation on* H_0 *is the same as that defined by* $(\Gamma \times S)|\, H_0$ (*Definition* 4.5).

Proof. Proposition 4.20 is an immediate consequence of Propositions 4.18 and 4.19.

V. Some Theorems on Virtual Groups

In Section V we first show that virtual subgroups of $\mathbb{R}^m \times \mathbb{Z}^k$ defined by proper free ergodic actions can be imbedded in Ω, the direct sum of countably many copies of $\mathbb{Z}/2\mathbb{Z}$ (Theorem 1). Next, we show that they can also be imbedded in $\mathbb{Z}$ (Theorem 2). Finally, we restrict our attention to virtual groups defined by proper free ergodic actions of $\mathbb{R}^m \times \mathbb{Z}^k$ on spaces (S, μ), where μ is finite and invariant. We will show that all such virtual groups are the same (Theorem 3).

In Section V we assume Γ is a locally compact separable group with properties 3.1 and 3.2 acting in a proper free ergodic fashion on (S, μ) a standard measure space. E and ν are chosen as in Proposition 2.10 and Definition 2.11, respectively. The directed sequence of liftable idempotents $\{h_{n,m}\}$ is chosen as in Proposition 4.20. Our immediate aim is to show how the equivalence relation associated with $\{f_{n,m}\}$ (Definition 4.5) can be described as the equivalence relation defined by a free action of Ω on a Borel set B conull in $H_0 = \mathrm{Dom}\, h_{0,1}$.

Let Ω_n be the nth copy of $\mathbb{Z}/2\mathbb{Z}$. Let Ω_n be generated by τ_n.

Definition 5.1. A Borel Ω-space H ([1, p. 328]) is said to "admit a sequence of dichotomies" if there are Borel sets H_n, $n = 0, 1,\ldots$, such that:

(a) $H = H_0$ and $H_n \supsetneqq H_{n+1}$, for all n, and

(b) H_n is the disjoint union of H_{n+1} and $\tau_{n+1} \cdot H_{n+1}$.

Proposition 5.2. *If the Borel* Ω-*space* H *admits a sequence of dichotomies and* $m > 0$, H_n *is the disjoint union of the sets*

$$\Big\{\tau(H_{n+m}) : \tau \in \bigoplus_{r=n+1}^{n+m} \Omega_r\Big\}.$$

Proof. A trivial induction on m.

PROPOSITION 5.3. *If the Borel Ω-space H admits a sequence of dichotomies, the action of Ω on H is free.*

Proof. A trivial consequence of Proposition 5.2.

5.4. *The Action of Ω*

We define an action of Ω on $H_0 = \operatorname{Dom} h_{0,1}$ as follows:

Since $h_{n,n+1}$ is $2:1$, $h_{n,n+1} \mid (H_n - H_{n+1})$ is $1:1$ and $h_{n,n+1}(H_n - H_{n+1}) = H_{n+1}$. We define the action of τ_{n+1} on $H_n - H_{n+1}$ to be $h_{n,n+1})$ and on H_{n+1} to be $(h_{n,n+1} \mid (H_n - H_{n+1}))^{-1}$. This extends to a unique action of Ω on H_0.

PROPOSITION 5.5. *The action of Ω defined in* 5.4 *admits a sequence of dichotomies* (*Definition* 5.1).

Proof. Trivial.

PROPOSITION 5.6. *The action of Ω preserves the measure class $[\nu \mid H_0]$.*

Proof. If $\tau \in \Omega$, the action of τ is by means of a liftable map.

PROPOSITION 5.7. *The action of Ω on H_0 defines the same equivalence relation as $\Gamma \times S \mid H_0$.*

Proof. By Proposition 4.20, the equivalence relation associated with the directed sequence of idempotents $\{h_{n,m}\}$ on H_0 is the same as that defined by $\Gamma \times S \mid H_0$.

If $\tau \in \Omega$, $\tau \in \bigoplus_{n=1}^m \Omega_n$ for some m, so $h_{0,m}(x) = h_{0,m}(\tau \cdot x)$ for all $x \in H_0$.

Conversely, by a trivial induction on m, $h_{0,m}(x) \in \Omega \cdot x$, and so, if $h_{0,m}(y) = h_{0,m}(x)$ for some m, $x \in \Omega \cdot y$.

PROPOSITION 5.8. *If the measure μ on the Γ-space S is finite and invariant, there is a finite measure $\lambda \sim (\nu \mid H_0)$ which is Ω-invariant.*

Proof. If $\tau \in \Omega$, τ acts by means of a liftable map. By Proposition 2.12, if A is any Borel set in H_0, $\nu(\tau \cdot A) \leqslant D\, \nu(A)$. This holds for all $\tau \in \Omega$. Thus

$$D^{-1} \leqslant \frac{d(\tau^*\nu)}{d\tau}(x) \leqslant D \text{ a.e.} \qquad \text{for all} \quad \tau \in \Omega.$$

Let

$$f(x) = \limsup_{m\to\infty} 2^{-m} \Sigma \left\{ \frac{d(\tau^*\nu)}{d\nu}(x) : \tau \in \bigoplus_{n=1}^{m} \Omega_n \right\}.$$

Then f is Borel and $D^{-1} \leqslant f(x) \leqslant D$ a.e. Also, if $\sigma \in \Omega, f(\sigma \cdot x)d(\sigma^*\nu)/d\nu = f(x)$, and so we define λ by $\lambda(A) = \int_A f(x)\, d\nu(x)$.

THEOREM 1. *Let Γ be a locally compact separable group with properties* 3.1 *and* 3.2 (e.g., $\mathbb{R}^n \times \mathbb{Z}^m$). *Let Γ act in a proper free ergodic fashion on (S, μ). Then the virtual subgroup of Γ, so defined, is isomorphic to a virtual subgroup of Ω defined by a proper free ergodic action of Ω on some (H_0, λ). The Ω-space H_0 admits a sequence of dichotomies (Definition* 5.1).

Furthermore, if μ is Γ-invariant, we may assume λ is Ω-invariant.

Proof. The action of Ω on H_0 is as in 5.4. If μ is not Γ-invariant, $\lambda = \nu \mid H_0$. If it is Γ-invariant, λ is chosen as in Proposition 5.8. Thus we have only to show that the ergodic groupoids ($\Gamma \times S$, [Haar $\times\, \mu$]) and ($\Omega \times H_0$, [Counting $\times\, \lambda$]) are similar. By Proposition 2.13, ($\Gamma \times S$, [Haar $\times\, \mu$]) is similar to ($\Gamma \times S \mid E$, [Counting $\times\, \nu \mid E$]). $\nu(H_0) > 0$, and so by the theorem of Arlan Ramsay mentioned in 1.18 ([3, Theorem 6.17 p. 290]), ($\Gamma \times S \mid E$, [Counting $\times\, \nu \mid E$]) is similar to ($\Gamma \times S \mid H_0$ [Counting $\times\, \lambda$]). By Proposition 5.7 and the freedom of the actions of Γ and Ω, we see that ($\Gamma \times S \mid H_0$, [Counting $\times\, \lambda$]) is the same ergodic groupoid as ($\Omega \times H_0$, [Counting $\times\, \lambda$]). This completes the proof.

PROPOSITION 5.9. *Given an ergodic action of Ω on (H, λ) admitting a sequence of dichotomies (Definition* 5.1), *there is a Borel set K, conull in H, and an ergodic action of Z on $(K, \lambda \mid K)$ such that the Ω-orbits and Z-orbits in K are the same.*

Furthermore, if λ is Ω-invariant, λ is also $\mathbb{Z}$-invariant.

Proof. We have a sequence $H = H_0 \supseteq H_1 \supseteq \cdots$ as in Definition 5.1.

The Borel set $\bigcap_n H_n$ meets each orbit in at most one point and so is null. Similarly, the set $\bigcap_n (\tau_n\tau_{n-1} \cdots \tau_1) \cdot H_n$ is null.

$H - \bigcap H_n$ is the disjoint union of the sets $(H_n - H_{n+1})$, $n \geqslant 0$.

$H - \bigcap (\tau_n\tau_{n-1} \cdots \tau_1) \cdot H_n$ is the disjoint union of the sets $(\tau_n\tau_{n-1} \cdots \tau_1) \cdot H_{n+1}$.

We define the action of $\mathbb{Z}$ thus: if $x \in H_n - H_{n+1} = \tau_{n+1}(H_{n+1})$, $1 \cdot x = (\tau_{n+1}\tau_n \cdots \tau_1) \cdot x$. Thus $(1\, \cdot)$ maps $H_n - H_{n+1}$ 1 : 1 onto $(\tau_n \cdots \tau_1) \cdot H_{n+1}$ and preserves the measure class of λ. If λ is Ω-invariant, λ will be Z-invariant. The action of $\mathbb{Z}$ is only defined on the conull set $K = \bigcap_n n \cdot (H - \bigcap_m H_m)$.

It is trivial that the $\mathbb{Z}$-orbits are contained in the Ω-orbits. By an obvious induction on m, the orbits of $(\bigoplus_{n=1}^m \Omega_n)$ in K are contained in the $\mathbb{Z}$-orbits. So the $\mathbb{Z}$-orbits are just the intersection of the Ω-orbits with K.

THEOREM 2. *Let Γ be a locally compact separable group with properties 3.1 and 3.2 (e.g., $\mathbb{R}^n \times \mathbb{Z}^m$). Let Γ act in a proper free ergodic fashion on (S, μ). The virtual subgroup of Γ so defined is isomorphic to a virtual subgroup of $\mathbb{Z}$ on some (K, λ).*

If μ is Γ-invariant, we may assume λ is $\mathbb{Z}$-invariant.

Proof. Theorem 2 is an immediate consequence of Theorem 1, Proposition 5.9, and the theorem of Arlan Ramsay discussed in 1.18.

We now restrict our attention to the case of a Γ-invariant finite measure μ. The Ω-space (H, λ) is as in Theorem 1 and λ is Ω-invariant. We may assume $\lambda(H) = 1$.

Since H admits a sequence of dichtomies, we have $H = H_0 \supseteq H_1 \supseteq \cdots$ as in Definition 5.1.

From Definition 5.10 to Proposition 5.14, we will be reexpressing some results of A. M. Vershik [9] in a different notation.

DEFINITION 5.10. Let $P_m : L^2(H, \lambda) \to L^2(H, \lambda)$ be defined by

$$(P_m f)(x) = 2^{-m} \Sigma \left\{ f(\sigma \cdot x) : \sigma \in \bigoplus_{n=1}^{m} \Omega_n \right\}.$$

PROPOSITION 5.11. *P_m is an orthogonal projection and if $m > n$, $P_m \leqslant P_n$.*

Proof. λ is Ω-invariant.

PROPOSITION 5.12. *If $f \in L^2(H, \lambda)$, $P_m f$ converges strongly in $L^1(H, \lambda)$ to the constant $\int_H f(x)\, d\lambda(x)$.*

Proof. Since P_m is a decreasing sequence of orthogonal projections, $P_m f$ converges to g, say, in $L^2(H, \lambda)$. $\lambda(H) = 1$, so $P_m f$ converges to g in $L^1(H, \lambda)$.

$(P_m f)$ is $(\bigoplus_{n=1}^m \Omega_n)$-invariant and so g is Ω-invariant as a point in $L^1(T, \lambda)$. Since Ω is countable, g is a.e. Ω-invariant. Since Ω acts ergodically, g is a.e. constant. $\int_H (P_m f)\,(x)\, dx = \int_H f(x)\, d\lambda(x)$ for all m and so $g = \int_H f(x)\, d\lambda(x)$ a.e. Thus $P_m f$ converges strongly to $\int_H f(x)\, d\lambda(x)$.

PROPOSITION 5.13. *Given disjoint Borel sets $X_1 \cdots X_k \subseteq H$ and $\epsilon > 0$, there is an integer m and a new, free action, $*$, of $\Phi_m = (\bigoplus_{n=1}^m \Omega_n)$ on H such that:*

(a) *the orbits $\Phi_m \cdot x$ and $\Phi_m{}^* x$ are the same, and*

(b) *every X_r can be approximated to within ϵ by a finite union of sets $\tau^* H_m$ (such that $\tau \in \Phi_m$).*

Proof. Choose k so that $2^{-k} \leqslant \epsilon/5$. Let $Y_r = X_r - H_k$. Let χ_r be the characteristic function of Y_r.

By Proposition 5.12, $(P_m \chi_r) \to \lambda(Y_r)$ in $L^1(H, \lambda)$. Choose $m \geqslant k$ so that $\| P_m \chi_r - \lambda(Y_r)\|_1 \leqslant \epsilon^2/25$, for $r = 1 \ldots L$.

The Φ_m-invariant set $B = \{x \in H: |(P_m \chi_r)(x) - \lambda(Y_r)| > \epsilon/5\}$ is of measure $< \epsilon/5$.

Let L_r be be largest integer such that $L_r \leqslant 2^m (P_m \chi_r)(x) = \#(Y_r \cap \Phi_m \cdot x)$ and so, for all $x \notin B$, $\#(Y_r \cap \Phi_m \cdot x) \geqslant L_r$.

H_m is a cross-section for the action of Φ_m, so we can find Borel maps $\sigma_1{}^r, \ldots, \sigma_{L_r}^r : (H_m - B) \to \Phi_m$ such that the maps $s_j{}^r$ defined by $s_j{}^r(x) = \sigma_j{}^r(x) \cdot x$ map $(H_m - B)$ isomorphically onto L_r disjoint subsets of Y_r.

We can also choose $(2^m - 1 - \sum_{r=1}^{R} L_r)$ Borel maps $\sigma_j{}^0 : (H_m - B) \to \Phi_m$ such that if $s_j{}^0(x) = \sigma_j{}^0(x) \cdot x$, the $s_j{}^0$ map $(H_m - B)$ isomorphically onto disjoint sets in $(H - B) - (H_m - B) - \bigcup_{r=1}^{R} \bigcup_{j=1}^{L_r} s_j{}^r(H_m - B)$.

Thus we have $2^m - 1$ Borel maps $s_j{}^r$ mapping $(H_m - B)$ isomorphically onto disjoint sets in $(H - B)$ whose union is $(H - B) - H_m$. Label the $s_j{}^r$ by elements of $\Phi_m - \{\mathrm{id}\}$ thus: $\{s_j{}^r\} = \{u_\tau : \tau \in (\Phi_m - \{\mathrm{id}\})\}$. Let $u_{\mathrm{Id}} = \mathrm{id}$. So $(H - B) = \bigcup_\tau u_\tau(H_m - B)$.

We define the action $*$ on $(H - B)$ thus: if $x = u_\tau(y)$, for some $y \in (H_m - B)$ and $\sigma \in \Phi_m$, $\sigma^* x = u_{\sigma\tau}(y)$. Define the action $*$ of Φ_m on B to be the old action.

Obviously, the new orbits are contained in the old and have 2^m points, so the old and new orbits are the same. Therefore H_m is a cross-section for the action $*$.

Given X_r, we can find $\tau^1, \ldots, \tau^{L_r} \in \Phi_m$ such that $u_{\tau^j} = s_j{}^r$ and so $\tau^j * H_m = s_j{}^r(H_m - B) \cup \tau^j \cdot (B \cap H_m)$. $s_j{}^r(H_m - B) \subseteq Y_r \subseteq X_r$ and the $s_j{}^r(H_m - B)$ are disjoint, so $\lambda(X_r - \bigcup_j s_j{}^r(H_m - B)) \leqslant \epsilon/5 + \lambda(Y_r) - L_r \lambda(H_m - B) \leqslant \epsilon/5 + \lambda(Y_r) - L_r \lambda(H_m) + L_r \lambda(B \cap H_m) \leqslant \epsilon/5 + \lambda(Y_r) - (L_r 2^{-m})(1 - \epsilon/5) \leqslant 2\epsilon/5 + \lambda(Y_r) - (L_r)2^{-m}$. Now $L_r + 1 \geqslant 2^{-m}((P_m \chi_r)(x) - \epsilon/5)$ if $x \notin B$ and so $L_r + 1 \geqslant 2^{-m} + \lambda(Y_r) - 2\epsilon/5$. Therefore $\lambda(X_r - \bigcup_j s_j{}^r(H_m - B)) \leqslant 4\epsilon/5$. Also

$$\lambda\left(\bigcup_{j=1}^{L_r} \tau^j(B \cap H_m)\right) \leqslant \lambda(B) \leqslant \epsilon/5.$$

Therefore X_r is approximated to within ϵ by $\bigcup_{j=1}^{L_r} \tau^j * (H_m)$.

PROPOSITION 5.14. *Let (H, λ) and (H^2, λ^2) be ergodic Ω-spaces with invariant probability measures admitting sequences of dichotomies (Definition 5.1). The equivalence relations defined by the actions are isomorphic in the sense that there is an essentially* $1:1$ *onto Borel map $f: H \to H^2$ such that $f^*\lambda = \lambda^2$ and for almost all x and $y \in H$, $x \in \Omega \cdot y$ iff $f(x) \in \Omega \cdot f(y)$.*

Proof. Choose a sequence of successively finer partitions $\mathbf{P}_N$ of H into 2^N Borel sets such that $\bigcup_N \mathbf{P}_N$ generates all Borel sets. Let $\mathbf{P}_N = \{X_1^N,..., X_{R_N}^N\}$. Choose a sequence $\epsilon_N \to 0$.

Suppose we have found an integer m_N and an action $*$ of $\Phi_{m_N} = \bigoplus_{n=1}^{m_N} \Omega_n$ as in Proposition 5.13, with ϵ_N in place of ϵ and $X_1^N,..., X_{R_N}^N$ in place of $X_1,..., X_R$. We can pull back the partition $\mathbf{P}_{N+1}$ to H_{m_N}, i.e., consider the partition generated by the 2^{m_N} partitions $\{\tau^*(X_r^N \cap \tau^* H_{m_N}): r = 1,..., R_N\}$. We replace ϵ by $\epsilon_{(m_N+1)} 2^{-m_N}$ and Ω by $\bigoplus_{n=m_N+1}^{\infty} \Omega_n$ and apply Proposition 5.13. Thus we obtain an action $*$ of $\bigoplus_{n=m+N1}^{m'} \Omega_n$ on H_{m_N} and hence an action $*$ of $\Phi_{m_{N+1}} = \bigoplus_{n=1}^{m_{N+1}} \Omega_n$ on H, where $m_{N+1} = m'$.

In this way we obtain an action $*$ of Ω on H such that:

(a) if $x \in H$, $\Omega^* x = \Omega \cdot x$, and

(b) $\{\tau * H_m : \tau \in \Phi_m\}$ forms a sequence of successively finer partitions of H into 2^m Borel sets of equal measure, whose union is dense in (H, λ).

We do the same for (H^2, λ^2). f is defined to be the unique (mod 0) Borel map taking $\tau * H_m$ to $\tau * H_m{}^2$ for all m and for all $\tau \in \Phi_m$.

If $\sigma \in \Omega$, $(\sigma^*) \circ f \circ (\sigma^*)$ also takes $\tau * H_m$ to $\tau * H_m{}^2$ for all $\tau \in \Phi_m$ and so $(\sigma^*) \circ f \circ (\sigma^*) = f$ a.e. Thus (H, λ) and (H^2, λ^2) are isomorphic as Ω-spaces (with the actions $*$) and so the corresponding equivalence relations are isomorphic.

THEOREM 3. *Let Γ_1 and Γ_2 be locally compact separable groups with properties* 3.1 *and* 3.2 (e.g., $\mathbb{R}^n \times \mathbb{Z}^m$). *Let Γ_i act in a proper free ergodic fashion on (S_i, μ_i), where μ_i is Γ_i-invariant and finite. The virtual subgroups of Γ_1 and Γ_2, so defined, are the same.*

Proof. Theorem 3 is an immediate consequence of Theorem 1 and Proposition 5.14.

PROPOSITION 5.15. *Assume Γ is a countable group acting in a proper free ergodic fashion on (S, μ), where μ is Γ-invariant. Given any two Borel sets A and B of equal finite measure, in S, there is a liftable isomorphism $f: A \leftrightarrows B$.*

Proof. As in Proposition 4.14, we can find a maximal (mod 0) pair (X, f) such that $X \subseteq A$ and f maps X isomorphically onto some subset of B. $\lambda(X) = \lambda(f(X))$, therefore if $\lambda(A - X) > 0$, $\lambda(B - f(X)) > 0$, Γ acts ergodically so $\Gamma \cdot (A - X) = S(\text{mod } 0)$ and since Γ is countable, there exists $\alpha \in \Gamma$ mapping a subset of $(A - X)$ of positive measure into $B - f(X)$. This contradicts the maximality of (X, f). Thus $\lambda(A - X) = 0$, so $\lambda(B - f(x)) = 0$.

In Theorem 3, we showed that in the measure-invariant case, the ergodic groupoids $\Gamma_i \times S_i$ are similar. We now prove a stronger result for finitely generated abelian groups (e.g. $\mathbb{Z}^k$).

The next theorem is a special case of a result proven by H. A. Dye [11, 12] for arbitrary countable abelian groups. R. M. Belinskaya [8] proved the special case $\Gamma = \mathbb{Z}$.

THEOREM 4. *Let Γ_i, $i = 1, 2$, be two finitely generated abelian groups acting in proper free ergodic fashion on (S_i, μ_i), where μ_i is Γ_i-invariant and $\mu_i(S_i) = 1$.*

Then the ergodic equivalence relations defined by these actions are isomorphic in the sense that there exists an essentially 1 : 1 *onto Borel map $f\colon S_1 \rightrightarrows S_2$ such that $f^*\mu_1 = \mu_2$ and $f(x) \in \Gamma_2 \cdot f(y)$ iff $x \in \Gamma_1 \cdot y$.*

Proof. Consider one such ergodic action and drop the subscript i. Γ is the product of a finite abelian group and $\mathbb{Z}^k$, so from Propositions 3.5. and 3.4, Γ has properties 3.1 and 3.2. Δ is a finite set in Γ. So if ν is chosen as in Definition 2.11 and E as in Proposition 1.10, $\nu(A) = c\mu(A)$ for all Borel sets $A \subseteq E$ where $c = \# \Delta$.

In Proposition 4.17, (b) may be replaced by (b′): $\nu(\text{Dom}\, g) = (1 - \epsilon)$ $\nu(E)$ simply by restricting the range of g.

Consider Proposition 4.18 and its proof. From Proposition 4.17(b′), $(g_{0,n}^* \nu)$ $(\text{Rang}\, g_{0,n} - \text{Dom}\, g_{n,n+1}) = \epsilon_n (g_{0,n}^* \nu)$ $(\text{Dom}\, g_{0,n})$, so by the argument of section (B) of the proof,

$$\nu(F_0) = \left(\prod_{n=1}^{\infty} (1 - \epsilon_n) \right) \nu(E).$$

Thus, given any positive $\delta < c^{-1} \nu(E)$, we can ensure that $\mu(F_0) = \delta$ by choosing ϵ_n so that $\prod (1 - \epsilon_n) = c\delta$.

We denote by $F_0{}^i$ the F_0 corresponding to the action of Γ_i on (S_i, μ_i) and we find some integer K such that $2^K > c_i$, $i = 1, 2$, so we can ensure that $\mu_1(F_0{}^1) = \mu_2(F_0{}^2) = 2^{-K}$.

As in the proof of Theorem 1, we define actions of Ω on H^1, H^2 conull in F_0^1, F_0^2, respectively.

We use Proposition 5.15 to define free actions of $(\mathbb{Z}/2\mathbb{Z})^K$ on S_i by maps liftable to Γ_i, for which the H^i are cross-sections. Thus we obtain ergodic actions of Ω on the S_i admitting a sequence of dichotomies (Definition 5.1). The Ω-orbits are contained in the Γ_i-orbits and, when restricted to the sets H^i, are the same. From the ergodicity of the actions of the Γ_i and the positivity of the $\mu(H^i)$, we see that the equivalence relation defined by the action of Ω is the same as that defined by the action of Γ_i. Theorem 4 now follows from Proposition 5.14.

VI. Some Less Abstract Results

In Section IV, we obtained various results about virtual groups. In Section V we interpreted these results in a somewhat less abstract setting.

We use George W. Mackey's generalization of the concept of a "flow built under a function" [7, Section 6]. If Γ_1 acts ergodically on $(S, [\mu])$ and $\phi: \Gamma_1 \times S \to \Gamma_2$ is a homomorphism of ergodic groupoids, we obtain a Boolean Γ_2-space, which we denote by M_ϕ, and a corresponding Borel Γ_2-space $(\tilde{S}_2, \tilde{\mu}_2)$.

Proposition 6.1. *Let Γ_i act ergodically on (S_i, μ_i), $i = 1, 2$, where μ is a quasi-invariant probability measure.*

If the corresponding virtual groups are isomorphic,

(a) *there exists a homomorphism $\phi: \Gamma_1 \times S_1 \to \Gamma_2$ such that there is a Γ_2-equivariant Borel isomorphism $f: S_2 \simeq \tilde{S}_2$ such that $f^*\mu_2 \sim \tilde{\mu}_2$,*

(b) *if $\Gamma_1 = \Gamma_2 = \mathbb{Z}$ and μ_1 is invariant, $\tilde{\mu}_2 \sim \hat{\mu}_2$, a σ-finite invariant measure.*

Proof. (a) We have $\alpha: \Gamma_1 \times S_1 \to \Gamma_2 \times S_2$ one half of the similarity Let $\psi: \Gamma_2 \times S_2 \to \Gamma_2$ be the homomorphism $(\alpha, x) \to \alpha$. From Theorem 7.11 on p. 299 of [3], $M_{\psi\circ\alpha}$ and M_ψ are isomorphic Boolean Γ_2-spaces.

From Theorem 7.10 on p. 299 of [3], M_ψ and M_2, the Boolean Γ_2-space defined by the Borel Γ_2-space (S_2, μ_2), are isomorphic.

Therefore, by Theorem 2, p. 333 of [1], there exists a unique (mod 0) Borel isomorphism $f: S_2 \simeq \tilde{S}_2$ such that $f^*: M_{\psi\circ\alpha} \simeq M_2$. Thus if $\phi = \psi \circ \alpha$, (a) is proved.

(b) In the construction of M_ϕ, $\Gamma_2 \times \Gamma_1$ acts on $\Gamma_2 \times S_1$ by

$(\alpha, \beta)\ (\gamma, s) = (\phi(\beta, s)\,\gamma\alpha^{-1},\ \beta \cdot s)$ and we have the measure $\lambda_2 \times \mu_1$ on $\Gamma_2 \times S_1$, where λ_2 is a Haar measure on Γ_2. There is a cross-section for the action of Γ_1 on $\Gamma_2 \times S_1$, namely $\{(\gamma, s)\colon 0 \leqslant \gamma < \phi(1, (-1) \cdot s)\}$. We identify this set with S_2 and define $\hat{\mu}_2$ to be the restriction of $\lambda_2 \times \mu_1$ to this set.

Remark 6.2. If $\hat{\mu}_2$ is the measure defined in Proposition 6.1(b), $\hat{\mu}_2$ will be finite, provided S_2 has a finite invariant measure in the measure class of μ_2.

Proof. $(df^*\mu_2)/d\hat{\mu}_2$ is invariant and Borel and so is constant.

PROPOSITION 6.3 (due to Larry Brown). *There are nonisomorphic proper virtual subgroups of the integers.*

Suppose $(S, \mu) = (\int_X S_x\, d\lambda(x), \int_X \mu_x\, d\lambda(x)$ *is a direct integral decomposition of* (S, μ) *into ergodic* $\mathbb{Z}$*-spaces* (S_x, μ_x), *where* $\mu_x(S_x) = 1$. *Let* $p\colon S \to X$ *be the corresponding projection. If* A *is a Borel set in* S, *let* $A_x = A \cap p^{-1}(x)$. *Then we have the following lemma.*

LEMMA. *If for all* x, $\mu_x \sim \tilde{\mu}_x$ *is an invariant* σ*-finite measure, then* $\mu \sim \tilde{\mu}$ *is an invariant* σ*-finite measure.*

Proof. Let $\{A_i\}_{i=1}^{\infty}$ be an algebra generating all Borel sets in S, then $\{A_{i,x}\}_{i=1}^{\infty}$ generates all Borel sets in S_x and so there is an i such that $0 < \tilde{\mu}_x(A_{i,x}) < \infty$. If A and B are Borel sets in S_x and $f\colon A \to B$ is a liftable 1 : 1 onto map, B is said to be a "copy" of A.

From Proposition 5.15, it follows that if $\tilde{\mu}_x(A_x) = \infty$ and $\tilde{\mu}_x(B_x) < \infty$, has infinitely many disjoint copies in A_x.

Therefore $\{x\colon \mu_x(A_{j,x}) = \infty\} = \{x$ s.t. $A_{k,x}$ has infinitely many copies in $A_{j,x}$ and $\mu_x(A_{k,x}) > 0\}$.

Given A and B Borel sets in S $\{x\colon B_x$ has at least N disjoint copies in $A_x\} = \bigcap_{m=1}^{\infty} \{x\colon$ for some n, some positive integers $k_1, \ldots, k_n$, and some integers $\alpha_1{}^r, \ldots, \alpha_n{}^r$, $r = 1, \ldots, N$,

(a) the sets $(A_{k_i})_x$ are disjoint,

(b) the sets $\alpha_i{}^r \cdot (B \cap A_{k_i})_x$ are disjoint,

(c) $\mu_x(B - \bigcup_{i=1} A_{k_i})_x) < 1/m\}$,

and so is a Borel set in X.

Therefore $\{x\colon \tilde{\mu}_x(A_{j,x}) = \infty\}$ is Borel in X. Since

$$\bigcup \{x : 0 < \tilde{\mu}_x(A_{i,x}) < \infty\} = X,$$

we can find a Borel set $A \subseteq S$ such that for all $x \in X$, $0 < \mu_x(A) < \infty$.

If E is any Borel set in S, $\mu_x(E)\,(\mu_x(A))^{-1} = \sup\{N/M$ s.t. N disjoint copies of A_x can be found in the union of M copies of $B_x\}$ and so an explicit formula for $\{x$ s.t. $N/M \leqslant \tilde{\mu}_x(E)\,(\tilde{\mu}_x(A))^{-1}\}$ can be found. It is similar to the one for $\{x$ s.t. B_x has at least N disjoint copies in $A_x\}$. Thus $\tilde{\mu}_x(E)\,(\tilde{\mu}_x(A))^{-1}$ is a Borel function of x. We define $\tilde{\mu}$ by $\tilde{\mu}(E) = \int_X \tilde{\mu}_x(E)\ (\mu_x(A))^{-1}\, d\lambda(x)$ and so obtain a σ-finite invariant measure $\tilde{\mu} \sim \mu$.

Proof of Proposition 6.3. Donald S. Ornstein has found an example of a $\mathbb{Z}$-space (S, μ) with no invariant σ-finite measure equivalent to μ.

[10].

It follows from the Lemma that there is an ergodic $\mathbb{Z}$-space (S, μ) with no invariant σ-finite measure equivalent to μ. Therefore the action of $\mathbb{Z}$ is proper and hence free.

By Proposition 6.1.(b), the virtual group so defined cannot be isomorphic to the one defined by an ergodic $\mathbb{Z}$-space (T, λ), where λ is a $\mathbb{Z}$-invariant finite measure.

THEOREM 5. *If Γ has properties* 3.1 *and* 3.2 *and Γ acts in a free proper ergodic fashion on $(S, [\mu])$, there is a (free) proper ergodic action of $\mathbb{Z}$ on some $(K, [\lambda])$ and a Borel function $f\colon K \to \Gamma$ defining a homomorphism $\phi\colon \mathbb{Z} \times K \to \Gamma$ such that the Borel Γ-space S is isomorphic to the Borel Γ-space $(\tilde{S}, [\tilde{\mu}])$ defined as the image of ϕ. The isomorphism takes $[\mu]$ to $[\tilde{\mu}]$.*

If μ is Γ-invariant and finite, we may assume $(K, [\lambda])$ to be $(K_0, [\lambda_0])$ a $\mathbb{Z}$-space chosen once and for all with λ_0 a $\mathbb{Z}$-invariant standard measure (e.g., *the action of $\mathbb{Z}$ on the circle with Lebesgue measure, defined by rotation* 1).

Proof. Theorem 5 follows from Theorems 2 and 3 and Proposition 6.1.

Remark 6.4. Theorem 5 is rather unsatisfactory. It seems possible that there are no Borel cross-sections for $\mathbb{Z}$-orbits in $\Gamma \times T$. The obstruction is that $\{\phi(n, t)\}_{n=1}^{\infty}$ could have accumulation points $t \in T$.

Replacing $f(t)$ by $f(t) - g(t) - g(1 \cdot t)$ gives rise to an isomorphic Γ-space, but it seems unlikely that this can be used to remove the accumulation points.

Construction 6.5. A directed sequence of Γ-spaces.

Suppose we are given an ergodic Ω-space (H, λ) admitting a sequence of dichotomies (Definition 5.1). Then we have Borel sets H_n as in Definition 5.1, and so we have a sequence of Γ-spaces $(\Gamma \times H_n)$, with the action $\alpha \cdot (\beta, x) = (\beta\alpha^{-1}, x)$.

Given any sequence of Borel functions $\psi_n : H_n \to \Gamma$, we can define $f_{n,n+1} : \Gamma \times H_n \to \Gamma \times H_{n+1}$ thus: $f_{n,n+1} \mid \Gamma \times H_{n+1} = \text{id}$ and, if $x \in H_{n+1}$ and $\Omega_{n+1} = \{\text{id}, \tau,\} f_{n,n+1}(\alpha, \tau \cdot x) = (\psi_n(x)\alpha, x)$.

$f_{n,n+1}$ is a homomorphism of Borel Γ-spaces, it is surjective (mod 0) and $(f_{n,n+1})^*$ (Haar $\times$ $(\lambda \mid H_n)) \sim$ (Haar $\times$ $(\lambda \mid H_{n+1}))$. So we have a directed system of Γ-spaces.

6.6. Let $\mathbf{A}_n$ be the Boolean Γ-space defined by the Borel space $\Gamma \times H_n$. $(f_{n,n+1})_* \ \mathbf{A}_{n+1} \subseteq \mathbf{A}_n$. Therefore $(f_{1,n})_* \ A_n \subseteq \mathbf{A}_1$. Let $\mathbf{A} = \bigcap_{n=1}^{\infty} (f_{1,n})_* \mathbf{A}_n$. $\mathbf{A} \subseteq \mathbf{A}_1$ and is a Boolean Γ-space.

Proposition 6.7. *The directed system of Construction* 6.5 *has a projective limit.*

Proof. $\mathbf{A}_1$ is separable as a metric space, therefore $\mathbf{A}$ is separable. Therefore $\mathbf{A}$ is countably generated. Also, since $\mathbf{A} \subseteq \mathbf{A}_1$, $\mathbf{A}$ admits a faithful finite measure. Since $\mathbf{A}$ is countably generated, it must be the Boolean σ-algebra corresponding to some standard Borel space. By Theorem 1 on p. 330 of [1], $\mathbf{A}$ corresponds to a Borel Γ-space (S, μ). $\mathbf{A} \subseteq (f_{1,n})_* A_n$, and so by [3, Theorem 3.6, p. 272], we obtain Borel Γ-equivariant maps $f_{n,\infty} : \Gamma \times H_n \to S$ such that $(f_{n,\infty})_* = (f_{1,n})_*^{-1} \mid \mathbf{A}$. Thus $f_{n+1,\infty} \circ f_{n,n+1} = f_{n,\infty}$ a.e.

Suppose $g_n : \Gamma \times H_n \to S'$ are Γ-equivariant Borel maps such that $(g_n)^*$ (Haar $\times$ $\lambda \mid H_n) \sim \mu'$ and $g_{n+1} \circ f_{n,n+1} = g_n$ a.e., where (S', μ') is some other Γ-space with a quasi-invariant measure μ'. Let $\mathbf{A}'$ be the Boolean Γ-space defined by (S', μ'). $g_n \circ f_{1,n} = g_{n+1} \circ f_{1,n+1}$, so $g_{1*} : \mathbf{A}' \to \mathbf{A}_1$ has its image in $\mathbf{A}$. Therefore there is a Borel map $h: S \to S'$ such that, if we restrict the codomain of g_{1*} to $\mathbf{A}$, we obtain h_*. $g_n = h \circ f_{n,\infty}$, and so (S, μ) is the projective limit of the standard Γ-spaces $(\Gamma \times H_n$, [Haar $\times$ $\lambda \mid H_n$]).

Theorem 6. *Assume Γ is a locally compact separable group with properties* 3.1 *and* 3.2.

Any free proper ergodic Γ-space $(S, [\mu])$ is the projective limit of a directed sequence of Γ-spaces constructed as in Construction 6.5.

If $[\mu]$ *contains a finite* Γ*-invariant measure, we may take the action of* Ω *on* (H, λ) *to be the one defined thus*: $H = [0, 1]$, $\lambda =$ *Lebesgue measure*, $H_n = [0, 2^{-n})$, *and if* $\Omega_n = \{\text{id}, \tau_n\}$ $(\tau_n \cdot)| H_n$ is the map $x \to x + 2^{-n}$.

Proof. From Theorem 1 and Proposition 6.1 with $\Gamma_2 = \Gamma$, $\Gamma_1 = \Omega$, we know that $(S, [\mu])$ is isomorphic to the image of a homomorphism $\phi: \Omega \times H \to \Gamma$.

From Proposition 5.14, if μ is Γ-invariant, we can take the action of Ω on H to be any proper ergodic action with an invariant finite measure admitting a sequence of dichotomies (Definition 5.1).

Now $M_\phi \subseteq \mathbf{A}_1$ and consists of the Ω-invariant members of $\mathbf{A}_1$. $M_\phi = \bigcap_{m=1}^{\infty} \mathbf{B}_m$, where $\mathbf{B}_m$ consists of the $(\bigoplus_{n=1}^{m} \Omega_n)$-invariant members of $\mathbf{A}_1$.

Since H_n is a cross-section for the action of $(\bigoplus_{n=1}^{m} \Omega_n)$, we can identify $\mathbf{B}_n$ with $\mathbf{A}_n$, the Boolean Γ-space defined by the Borel Γ-space $\Gamma \times H_n$. The imbedding $\mathbf{B}_{n+1} \to \mathbf{B}_n$ then becomes $(f_{n,n+1})_* : \mathbf{A}_{n+1} \to \mathbf{A}_n$, where $f_{n,n+1}$ is defined as in Construction 6.5 with $\psi_{n+1}(s) = (\phi(\tau_n, s))^{-1}$.

Since M_ϕ is given to us in [7, Section 6], M_ϕ is certainly countably generated. If we identify each $\mathbf{B}_n$ with $\mathbf{A}_n$, M_ϕ becomes the $\mathbf{A}$ of 6.6. Thus the image of ϕ is the projective limit, as required.

VII. Actions Which Are Not Free

We are interested primarily in $\mathbb{R}^m \times \mathbb{Z}^k$, so we restrict our attention to abelian groups. Actions which are not free present neither difficulty nor any new points of interest. The situation is summed up in:

PROPOSITION 7.1. *Let* Γ *be an abelian locally compact separable group acting ergodically on* $[S, [\mu])$, *a standard space with an invariant measure-class. The stabilizer subgroup is almost everywhere constant (call this constant* Γ_0), *so* Γ/Γ_0 *acts freely and ergodically on* $(S, [\mu])$.

Proof. From [1, Lemma 2, p. 329], we can assume that S is the universal Γ-space with a metric ρ.

Given any set $E \subseteq \Gamma$, define $S(E) = \{s \in S\colon \text{for all } \alpha \in E\ \alpha \cdot s \neq s\}$. If Δ is compact, $S(\Delta) = \bigcup_n \{s \in S : \text{for all } \alpha \in \Delta, \rho(\alpha \cdot s, s) \geqslant n^{-1}\}$ and so is a countable union of closed sets and is thus Borel.

$S(\Delta)$ is Γ-invariant and so is either null or conull. Choose $\{\Delta_i\}_{i=1}^{\infty}$ a sequence of compact sets in Γ such that every open set in Γ is the union of some family of the Δ_i.

Let $\tilde{S} = S - \{S(\Delta_i)$ s.t. $S(\Delta_i)$ is null$\} - \{(S - S(\Delta_i))$ s.t. $S(\Delta_i)$ is conull$\}$. Then $\tilde{S}$ is conull in S and for all i, $\tilde{S} \cap S(\Delta) = \varnothing$ or $\tilde{S}$.

Given any open set $U \subseteq \Gamma$, U is the union of some family of the Δ_i, therefore $S(U) \cap \tilde{S} = \varnothing$ or S.

We claim the stabilizer of s is constant on S. Let $s_1, s_2 \in S$ have stabilizers Γ_1 and Γ_2, respectively; then $\tilde{S} \cap S(\Gamma - \Gamma_i) \neq \varnothing$ since $s_i \in S(\Gamma - \Gamma_i)$, so $\tilde{S} \cap S(\Gamma - \Gamma_i) = \tilde{S}$, for $i = 1, 2$. Therefore $s_2 \in S(\Gamma - \Gamma_1)$; thus if $\alpha \notin \Gamma_1$, $\alpha \cdot s_2 \neq s_2$, so $\alpha \notin \Gamma_2$. Similarly, if $\alpha \notin \Gamma_2$, $\alpha \in \Gamma_1$, therefore $\Gamma_1 = \Gamma_2$.

References

1. G. W. Mackey, Point realizations of transformation groups, *Illinois J. Math.* **6** (1962).
2. G. W. Mackey, The theory of group representations, Summer 1955, Notes by Dr. Fell and Dr. Lowdenslager, Department of Mathematics, University of Chicago.
3. A. Ramsay, Virtual groups and group action, *Advances in Math.* **6** (1971), 253–322.
4. J. L. Kelley, "General Topology," Van Nostrand.
5. P. R. Halmos, "Measure Theory," Van Nostrand.
6. W. Ambrose, representations of ergodic flows, *Ann. of Math.* **42** (1941), 723–739.
7. G. W. Mackey, Ergodic theory and virtual groups, *Math. Ann.* **166**, 187–207.
8. R. M. Belinskaya, Partition of Lebesgue space in trajectories defined by ergodic automorphisms, *Functional Anal. Appl.* 2 (1968), 190–199.
9. A. M. Vershik, Theorem on lacunary isomorphism of monotonic sequences of partitions, *Functional Anal. Appl.* 2 (1968), 200–203.
10. D. S. Ornstein, On invariant measures, *Amer. Math. Soc. Bull.* **66** (1960).
11. H. A. Dye, On groups of measure preserving transformations I, *Amer. J. Math.* **85** (1963), 551–576.
12. H. A. Dye, On groups of measure preserving transformations II, *Amer. J. Math.* **85** (1963), 551–576.